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AN INTRODUCTION
TO THE THEORY OF
AUTOMORPHIC FUNCTIONS

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AUTOMORPHIC FUNCTIONS

BY

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PREFACE

Owing largely to the researches of Poincaré and Klein the domain of Automorphic Functions has expanded enormously during the last thirty-five years; and the ramifications of the subject have extended into many and diverse fields. This has caused embarrassment in the selection of materials for a book of modest dimensions, and has necessitated a brief treatment, or in some cases the exclusion, of many important and attractive subjects. The aim throughout has been to present in as thorough a manner as possible the concepts and theorems on which the theory is founded, and to describe in less detail certain of its important developments.

The present tract had its origin in a series of lectures on Automorphic Functions given to the Mathematical Research Class of the University of Edinburgh during the Spring Term of 1915.

I wish to express a grateful acknowledgment of my indebtedness to Professor Whittaker, who has read the manuscript during the course of its preparation, and has made many valuable suggestions; and to Mr Herbert Bell, who has assisted in the preparation of the Bibliography.

L. R. F.

The University of Edinburgh,
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CHAPTER I

LINEAR TRANSFORMATIONS

1. The Linear Transformation.—If z is a complex quantity whose real part is x and whose imaginary part is iy, it is customary to represent z by a point in a plane whose abscissa is x and whose ordinate is y, the coordinates being referred to perpendicular axes. If x, or y, or both become infinite, the point recedes to an infinite distance from the origin of coordinates. This is expressed by $z=\infty$, and infinity is considered to be a single *point* in the complex plane. In this respect the conception of infinity differs from that employed in projective geometry, where the infinite region is considered to be a line.*

Consider $z'=f(z)$, where $f(z)$ is a function of z, and let the variable z' be represented on a second plane. To each point z for which the function $f(z)$ is defined there correspond one or more values of z' given by the above equation. To points, curves, and areas in the z-plane there correspond by virtue of the equation points, curves, and areas in the z'-plane.

The configurations in the one plane are said to be *transformed* into configurations in the other plane. The whole theory of automorphic functions depends upon a particular type of transformation, defined as follows:—

DEFINITION.—*The transformation*

$$z'=\frac{az+b}{cz+d} \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

where a, b, c, d *are constants* (*real or complex*) *and* $\mathrm{ad}-\mathrm{bc}\neq 0$ *is called a linear transformation.*†

* The reason for this is that the kind of transformations most frequently considered in the theory of functions of a complex variable transform the infinite region into a point in the finite part of the plane: whereas ordinary projection in geometry transforms the infinite region into a line.

† Also called a *linear substitution*, a *homographic transformation*, or a *homographic substitution*. If $ad-bc=0$, the equation reduces to $z'=$constant; but this case is without interest.

The quantity $ad-bc$ is called the *determinant* of the transformation. It will be convenient to have always

$$ad-bc=1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the determinant in the general case taking this value if the numerator and denominator of the fraction in the second member be divided by $\pm\sqrt{(ad-bc)}$.

Let equation (1) be solved for z.

$$z=\frac{-dz'+b}{cz'-a} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is called the *inverse* of the transformation (1). It is a linear transformation with the same determinant as (1). We state then—

THEOREM 1.—*The inverse of a linear transformation is a linear transformation.*

To each value of z there corresponds one, and only one, value of z' by (1); and conversely to each value of z' there corresponds one, and only one, value of z by (3). In other words, the z-plane is transformed into the z'-plane in a *one-to-one* manner.

It will be most serviceable to represent the values of z' not on a different plane, but on the same plane and with the same system of coordinates as are used for representing z. In all that follows but one plane will be used unless the contrary is explicitly stated. The results just found can be stated in the following—

THEOREM 2.—*The* z*-plane is transformed into itself in a one-to-one manner by a linear transformation.*

Let the points z' be subjected to a linear transformation,

$$z''=\frac{\alpha z'+\beta}{\gamma z'+\delta} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Expressing z'' as a function of z,

$$z''=\frac{\alpha[(az+b)/(cz+d)]+\beta}{\gamma[(az+b)/(cz+d)]+\delta}=\frac{(\alpha a+\beta c)z+\alpha b+\beta d}{(\gamma a+\delta c)z+\gamma b+\delta d} \quad . \quad (5)$$

Making the transformation (1) and then making (4) is equivalent to the single transformation (5). Now (5) is also a linear transformation; its determinant in the form in which the fraction is written is $(ad-bc)(\alpha\delta-\beta\gamma)$. It is worth noting that if the determinants of (1) and (4) are each unity, the determinant of

(5) is unity also without further change. If z'' be subjected to a linear transformation, we get again a linear transformation of z, and so on. Thus we have, in general—

THEOREM 3.—*The successive performance of any number of linear transformations is equivalent to a single linear transformation.*

The linear, or homographic, transformation has the property that *it leaves invariant the cross-ratio of four points.* For if z_1, z_2, z_3, z_4 are four points, and z_1', z_2', z_3', z_4' are the points into which they are transformed, we have, on substituting for z_1', z_2', z_3', z_4' their values from (1),

$$\frac{(z_1'-z_2')(z_3'-z_4')}{(z_1'-z_3')(z_2'-z_4')}=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} \quad . \quad . \quad . \quad (6)$$

2. The Fixed Points of the Transformation.—The points which remain unchanged under the transformation (1) are found by solving the equation

$$z=(az+b)/(cz+d), \quad \text{or} \quad cz^2+(d-a)z-b=0 \quad . \quad . \quad (7)$$

If $c \neq 0$, the fixed points are

$$\xi_1=\frac{a-d+\sqrt{[(a-d)^2+4bc]}}{2c}; \ \xi_2=\frac{a-d-\sqrt{[(a-d)^2+4bc]}}{2c} \quad . \quad (8)$$

If $c=0$, the fixed points are

$$\xi_1=\infty; \quad \xi_2=b/(d-a) \quad . \quad . \quad . \quad . \quad (9)$$

There is but one fixed point if $(a-d)^2+4bc=0$; or, when $ad-bc=1$, if $(a+d)^2=4$. It is the point

$$\xi=(a-d)/2c \, . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (7) will have more than two roots only in case it is identically satisfied; that is, if $c=d-a=b=0$. In this case every point is unchanged in position.

THEOREM 4.—*The only linear transformation with more than two fixed points is the identical transformation* $z'=z$.

By means of the last theorem we are able to establish the following proposition—

THEOREM 5.—*There is one, and only one, linear transformation which transforms three distinct points* z_1, z_2, z_3 *into three distinct points* z_1', z_2', z_3'.

We shall prove first that there is not more than one such trans-

formation. Let $z'=(az+b)/(cz+d)$, and $z''=(\alpha z+\beta)/(\gamma z+\delta)$ be two transformations carrying z_1, z_2, z_3 into z_1', z_2', z_3'. Eliminating z, we get z'' as a linear function of z', $z''=(Az'+B)/(Cz'+D)$. We know from Theorems 1 and 3, without actually performing the elimination, that z'' depends linearly upon z'; for z is derived from z', and z'' from z, by a linear transformation. When $z=z_1, z_2$, or z_3, we have by hypothesis $z''=z'=z_1', z_2'$, or z_3'. The transformation from z' to z'' has then three fixed points, z_1', z_2', z_3'; and by Theorem 4 $z''=z'$. The two transformations are thus the same, which proves that there is not more than one transformation with the desired properties.

That there is always one such transformation we prove by actually setting it up. If none of the six values is infinite, consider the transformation defined by

$$\frac{z'-z_1'}{z'-z_2'}\,\frac{z_3'-z_2'}{z_3'-z_1'}=\frac{z-z_1}{z-z_2}\,\frac{z_3-z_2}{z_3-z_1} \quad . \quad . \quad . \quad (11)$$

an equation which expresses the equality of the cross-ratios $(z'z_1', z_2'z_3')$ and (zz_1, z_2z_3). This is of the form (1) when solved for z' in terms of z. It obviously transforms z_1, z_2, z_3 into z_1', z_2', z_3'. The equation (11) depends upon both variables if each set of three points is distinct; the determinant of the transformation then does not vanish [see footnote, Section 1].

If $z_1=\infty$, $z_2=\infty$, or $z_3=\infty$, it is necessary to replace the second member of (11) by

$$\frac{z_3-z_2}{z-z_2}, \quad \frac{z-z_1}{z_3-z_1}, \quad \text{or} \quad \frac{z-z_1}{z-z_2}$$

respectively; and a similar change is necessary in the first member for an infinite value of z_1', z_2', or z_3'. In any case there is one transformation with the desired properties, and the theorem is established.

Equation (11) is a convenient form for use in actually setting up the transformation carrying three given points into three given points.

3. Conformal Transformations.—Let p be a point in a plane, and let C_1 and C_2 be two curves issuing from p. Let θ be the angle between the curves (*i.e.* the angle between their tangents at p). For convenience we shall consider the angle positive if in passing from a point on C_1 to a point on C_2 through the region

of the included angle the motion is counter-clockwise around p. Now let a transformation of the plane be made such that p, C_1, C_2 are transformed into a point p' and two curves C_1' and C_2' issuing therefrom. The transformed angle θ' will be that angle between C_1' and C_2' which includes the transforms of the points within the original angle θ. It is positive if in passing from C_1' to C_2' through the included angle the motion is counter-clockwise around p'. The cases with which we shall be chiefly concerned are those in which $\theta' = \pm\theta$.

DEFINITION.—*A transformation preserving the magnitude of angles is called a conformal transformation. It is directly or inversely conformal according as the sign of the angle is preserved or changed.*

It is a theorem in the elements of the Theory of Functions that a transformation of the complex plane $z' = f(z)$, where $f(z)$ is any analytic function of z, preserves both the magnitudes and signs of angles, except at the points where $f(z)$ has singularities.* Laying aside the two points $z = \infty$ and $z = -d/c$, the latter of which gives $z' = \infty$, since we have not defined the angle between lines meeting at ∞, we can state for reference the following—

THEOREM 6.—*The linear transformation is directly conformal.*

It is a theorem of the Theory of Functions, whose proof is beyond the scope of this tract, that the *most general* directly conformal transformation of the z-plane is of the form $z' = f(z)$, where $f(z)$ is an analytic function of z.† In general the correspondence of the points of the z-plane, and these of the z'-plane is not one-to-one. For example $z' = \sqrt{z}$ gives two values of z' for each value of z; $z' = \sin z$ gives infinitely many values of z for each value of z'. The question naturally arises whether the linear transformation is the most general one possessing the quality of uniqueness mentioned in Theorem 2. This is in fact the case.

For let $z' = f(z)$ be the most general function yielding a one-to-one transformation. Let $z = q$ be the point for which $z' = \infty$; $f(z)$ then has a pole at q and nowhere else. The function can be written in the form

$$z' = \frac{A_m}{(z-q)^m} + \frac{A_{m-1}}{(z-q)^{m-1}} + \cdots + \frac{A_1}{z-q} + A_0,$$

* See Forsyth, *Theory of Functions*, sec. 9. † *Ibid.*, sec. 253.

where m is the order of the pole. Clearing of fractions we get an equation of the m-th order in z. To each value of z' there are m values of z. Hence m must equal 1. Therefore $z' = \frac{A_1}{z-q} + A_0 = \frac{A_0 z + A_1 - A_0 q}{z-q}$, which is of the form (1). We have then established the following—

THEOREM 7.—*The most general one-to-one directly conformal transformation of the plane into itself is a linear transformation.*

4. Inversion in a Circle.—There is an intimate relation, as we shall see presently, between the linear transformation of the complex variable and the geometrical transformation of the plane known as "inversion in a circle."

Consider a circle with centre at O and of radius r. Inversion in the given circle is the transformation resulting from replacing each point P of the plane by a point P′ lying on the line OP and whose distance from O satisfies the equation $OP' . OP = r^2$. P and P′ are said to be *inverse* with respect to the circle. If O, P, P′ have the coordinates (f, g), (x, y), (x', y') respectively, the equations of the transformation are

$$x' - f = \frac{r^2(x-f)}{(x-f)^2 + (y-g)^2}, \quad y' - g = \frac{r^2(y-g)}{(x-f)^2 + (y-g)^2}.$$

If the centre O moves to infinity, the circle becomes a straight line. The point P′ approaches a position such that the line PP′ is perpendicular to and is bisected by the given line. That is, inversion in a straight line is a mere reflection in the line.

We state here for reference some well-known properties of inversions:—

(α) Angles are preserved in magnitude but changed in sign; that is, the transformation is inversely conformal.

(β) Circles are transformed into circles, the straight line being included as a special case of the circle.

(γ) Two points inverse with respect to a circle C are transformed into two points inverse with respect to the transformed circle C′.

The connection between inversions and linear transformations arises from (α) above. The inversion is a one-to-one transformation of the plane into itself which preserves the magnitude

of the angle but changes its sign. If now two inversions, or any even number, be made in succession, the result is a one-to-one transformation of the plane which preserves both the magnitudes and the signs of the original angles. According to Theorem 7, such a transformation is a linear transformation. Hence—

THEOREM 8.—*The successive performance of an even number of inversions is equivalent to a linear transformation of the plane into itself.*

The converse of this theorem will now be proved. The transformation

$$z' = (az+b)/(cz+d)$$

can be broken up into the succession of the following transformations:—

$$\begin{array}{lll} (a) & z_1 = z + \alpha, & \text{where } \alpha = d/c, \\ (b) & z_2 = e^{i\theta} z_1, & \text{where } c^2/(ad-bc) = \mathrm{A}e^{i\theta}, \quad \mathrm{A} > 0, \\ (c) & z_3 = \mathrm{A} z_2, & \\ (d) & z_4 = -1/z_3, & \\ (e) & z' = z_4 + \beta, & \text{where } \beta = a/c. \end{array}$$

That these are equivalent to the given transformation is verified by eliminating z_1, z_2, z_3, z_4. We shall show that each of the five transformations is equivalent to an even number of inversions, which proves that the given general linear transformation is equivalent to an even number of inversions.

(*a*) Putting $z = x + iy$, $z_1 = x_1 + iy_1$, $\alpha = a' + ia''$, the transformation, on breaking into real and imaginary parts, is

$$x_1 = x + a', \quad y_1 = y + a''.$$

This is a translation. Let $\mathrm{O}\alpha$ (fig. 1) be the line joining the origin to the point α. Each point of the plane is translated parallel to the line $\mathrm{O}\alpha$ a distance equal to the length $\mathrm{O}\alpha$. Let AB, CD be two lines perpendicular to $\mathrm{O}\alpha$ and at a distance $\frac{1}{2}\mathrm{O}\alpha$ apart. A re-

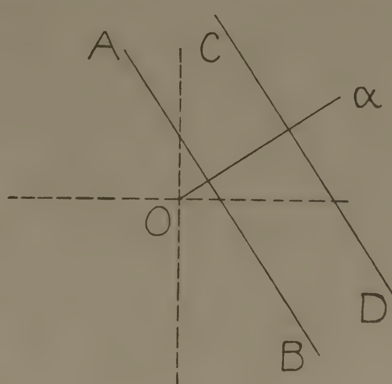

FIG. 1.

flection in AB (as in figure) followed by a reflection in CD is equivalent to the translation under consideration.* Thus (a) is equivalent to two inversions.

(b) Expressing $e^{i\theta}$ in the form $\cos\theta + i\sin\theta$, we get in this case, equating real and imaginary parts of the two members of the equation,

$$x_2 = x_1 \cos\theta - y_1 \sin\theta, \quad y_2 = x_1 \sin\theta + y_1 \cos\theta,$$

a rotation about the origin through an angle θ.

Let A′A and B′B be two lines through the origin such that the angle AOB is $\frac{1}{2}\theta$ (fig. 2). We verify at once that a reflection in A′A followed by a reflection in B′B is equivalent to the rotation in question. This case is thus reducible to a pair of inversions.

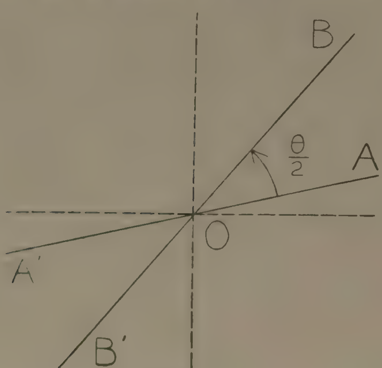

FIG. 2..

(c) This is the case of expansion from the origin

$$x_3 = \mathrm{A}x_2, \quad y_3 = \mathrm{A}y_2.$$

It is equivalent to the two inversions: (1) an inversion in a circle centre at the origin and radius r; followed by (2) an inversion in a circle centre at the origin and radius $r\sqrt{\mathrm{A}}$. For if P′, P″ are the successive positions of P, $\mathrm{OP}'' = \mathrm{A}r^2/\mathrm{OP}' = \mathrm{A}r^2/(r^2/\mathrm{OP}) = \mathrm{A}\,.\,\mathrm{OP}$.

(d) $$x_4 + iy_4 = \frac{-1}{x_3 + iy_3} = \frac{-x_3 + iy_3}{x_3^2 + y_3^2}$$

or $$x_4 = \frac{-x_3}{x_3^2 + y_3^2}, \quad y_4 = \frac{y_3}{x_3^2 + y_3^2}$$

This is equivalent to an inversion in the unit circle with centre at the origin

$$x_0 = \frac{x_3}{x_3^2 + y_3^2}, \quad y_0 = \frac{y_3}{x_3^2 + y_3^2},$$

* We note that in this and succeeding cases it is sufficient to verify that three points are transformed alike by the even number of inversions and by the linear transformation. This follows from Theorem 5.

In the present case we note that the points on AB are unchanged by a reflection in AB and are carried parallel to Oα a distance equal to the length Oα by a reflection in CD.

followed by a reflection in the imaginary axis

$$x_4 = -x_0, \quad y_4 = y_0.$$

(*e*) This, like (*a*), is a translation.

Combining these results, we get the converse of the preceding theorem—

THEOREM 9.—*Any linear transformation is equivalent to the successive performance of an even number of inversions.*

It is evident that the representation of a linear transformation as a succession of inversions is not unique; that the transformation can, in fact, be broken up into an even number of inversions in infinitely many ways.

The properties of inversions furnish us with valuable information concerning linear transformations. Thus (β) and (γ) preceding yield the following important theorem—

THEOREM 10.—*When the complex plane is subjected to a linear transformation,*

(*a*) *Circles are transformed into circles;*

(*b*) *If two points are inverse with respect to a circle, the transformed points are inverse with respect to the transformed circle.*

Since three points determine a circle, a given circle can be transformed into a given circle by setting up a transformation carrying three distinct points of the first circle into three distinct points of the second circle (Theorem 5). These points can be chosen in infinitely many ways, Hence—

THEOREM 11.—*There exist infinitely many linear transformations which transform a given circle into a second given circle.*

COROLLARY.—*There exist infinitely many linear transformations which transform a given circle into itself.*

5. The Transformed Elements of Length and of Area.—In the transformation

$$z' = (az + b)/(cz + d)$$

let z be given an increment dz; the transformed point z' is given an increment $dz' = (dz'/dz)dz$. The length of dz is $|dz|$, the absolute value or modulus of dz; and the length of dz' is $|dz'/dz|\,|dz|$. A small rectangle at the point z of sides dz_1 and dz_2 has the area $|dz_1 dz_2|$. Since angles are preserved, the

transformed area is a rectangle at z'. Its area is $|dz_1'dz_2'|$, or $|dz'/dz|^2|dz_1dz_2|$. Now

$$\frac{dz'}{dz}=\frac{ad-bc}{(cz+d)^2};$$

and if the determinant of the transformation, $ad-bc$, is unity, we can state—

THEOREM 12.—*If the determinant of the transformation is* 1, *elements of length in the neighbourhood of a point* z *are multiplied on transformation by* $|cz+d|^{-2}$; *and elements of area are multiplied by* $|cz+d|^{-4}$.

6. Types of Linear Transformations.—We have already noted a separation of linear transformations into two classes. Laying aside the identical transformation $z'=z$, a transformation may have either two fixed points or one fixed point. The number of fixed points and the behaviour of the transformation with reference to its fixed points furnish a useful basis of classification. We turn first to the larger class with two fixed points.

If in Equation (11) we put ξ_1 and ξ_2, the fixed points, for z_1 and z_2, the transformation becomes

$$\frac{z'-\xi_1}{z'-\xi_2}=K\frac{z-\xi_1}{z-\xi_2} \quad . \quad . \quad . \quad . \quad (12)$$

where $K=(z_3-\xi_2)(z_3'-\xi_1)/(z_3-\xi_1)(z_3'-\xi_2)$. K is called the *multiplier* of the transformation. It depends apparently upon the particular point z_3 chosen, but such is not the case; if for z_3, z_3' we put any other pair of corresponding points, we see by Equation (6) that K is unchanged.* The character of K is used to classify the transformations with two fixed points; and we shall see presently that it determines the behaviour of the transformation with reference to the fixed points.

It will simplify the problem to transform the plane in which z and z' are represented by a linear transformation. Let

$$Z=\frac{z-\xi_1}{z-\xi_2}, \quad Z'=\frac{z'-\xi_1}{z'-\xi_2} \quad . \quad . \quad . \quad . \quad (13)$$

* Considering the transformation in the form (1), Section 1, we note that when $z_3=\infty$, $z_3'=a/c$. Using these values, $K=(a-c\xi_1)/(a-c\xi_2)$, ξ_1 and ξ_2 having the values in Equations (9).

The transformation becomes

$$Z' = KZ \quad . \quad . \quad . \quad . \quad . \quad (14)$$

in which the fixed points ξ_1, ξ_2 are now shifted to 0 and ∞. Having studied this latter transformation, the results can be carried back by the inverse of (13) to the plane of z, z'.

Writing K in terms of its modulus A (>0) and its amplitude θ, so

$$K = Ae^{i\theta},$$

we distinguish the following cases:—

(*a*) *The hyperbolic* transformation, $K = A$.*

The transformation $Z' = AZ$ is the expansion from the origin studied in Section 4 (*c*). We observe at once the following facts concerning it: (1) a straight line through the origin (that is, a circle through the fixed points 0, ∞) is transformed into itself; (2) any circle with centre at the origin (and hence orthogonal to the family of fixed lines) is transformed into some other circle with its centre at the origin.

Let the ZZ'-plane be now transformed back into the zz'-plane by means of Equations (13). The points 0 and ∞ become ξ_1 and ξ_2 respectively. The straight lines through the origin become circles through the points ξ_1 and ξ_2; and the circles with centre at the origin become, since angles are preserved, circles orthogonal to the family through ξ_1 and ξ_2.

In the hyperbolic transformation then (1) any circle through the fixed points is transformed into itself; (2) any circle orthogonal to the family of fixed circles is transformed into some other circle orthogonal to the fixed circles. We note also that, since 0 and ∞ are inverse points with respect to any circle with centre at the origin, the points ξ_1 and ξ_2 are, applying Theorem 10 (*b*), inverse points with respect to any circle orthogonal to the fixed circles.

Fig. 3 shows the two families of circles just mentioned. The way in which a region is transformed is indicated, each shaded region being transformed into the next in the direction of the arrow.

* The names *elliptic*, *hyperbolic*, *parabolic*, were first used in this connection by F. Klein in 1878. They were suggested by the connection of substitutions of these types with displacements in absolute geometry of the elliptic, hyperbolic, and parabolic type.

(*b*) *The elliptic transformation*, $K = e^{i\theta}$.

The transformation $Z' = e^{i\theta}Z$ is the rotation about the origin discussed in Section 4 (*b*). The circles of the preceding case play a part here, but their rôles are changed. The circle with centre at the origin is transformed into itself; a line through the origin is transformed into another line through the origin, making an angle θ with the given line; the points 0 and ∞ are inverse points with respect to the fixed circles.

Transferring these results to the zz'-plane: (1) a circle through

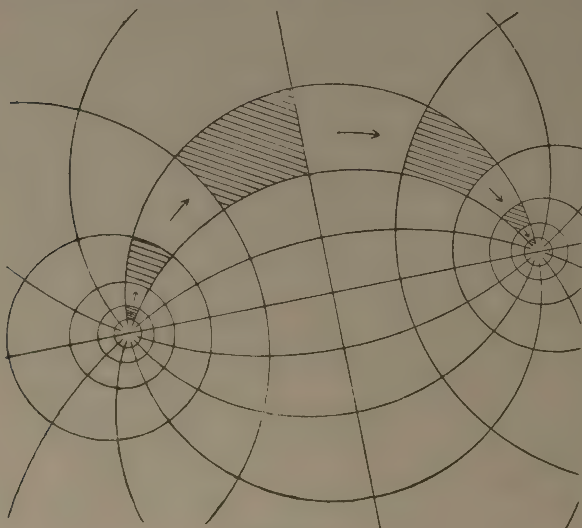

FIG. 3.

the fixed points is transformed into a circle through the fixed points intersecting the given circle at an angle θ; (2) the circles orthogonal to the circles through the fixed points are the fixed circles; (3) the fixed points are inverse points with respect to each of the fixed circles.

The behaviour of the transformation is shown in fig. 4. Here $\theta = \frac{1}{3}\pi$, and only one of the fixed points is shown. Each shaded region is transformed into the next in the direction of the arrow.

(*c*) *The loxodromic* transformation*, $K = Ae^{i\theta}$.

This transformation, $Z' = Ae^{i\theta}Z$, is a combination of the preced-

* The term *loxodromic* was introduced by F. Klein in 1882.

ing ones. It involves both a stretching from the origin and a rotation about the origin. In general there are no fixed circles; but in the particular case $\theta=\pi$, a stretching from the origin followed by a rotation through two right angles, each line through the origin is transformed into itself, the half of the line on the one side of the origin being transformed into the half on the other side.

In the zz'-plane, then, the loxodromic transformation has in general no fixed circles. In the particular case $\theta=\pi$ each circle through the fixed points is transformed into itself, the two

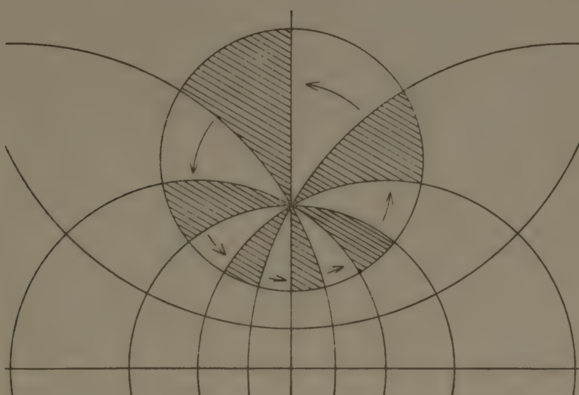

FIG. 4.

parts into which the circumference is divided by the fixed points being interchanged.

We note an essential difference between this particular case and the two preceding types with reference to the fixed circles. In the elliptic and hyperbolic transformations the interior of the fixed circle is transformed into itself; in the loxodromic transformations in which there are fixed circles, the interior of the fixed circle is transformed into its exterior.

(*d*) *The parabolic transformation, one fixed point.*

When determining the fixed points we found that if $(a-d)^2+4bc=0$ the transformation has but one fixed point, namely $\xi=(a-d)/2c$. In order to express the transformation in terms of the fixed point we note that the points $z=\infty$, $-d/c$ are transformed into $z'=a/c$, ∞. Setting up the unique transforma-

tion carrying ∞, ξ, $-d/c$ into a/c, ξ, ∞, we can write the transformation in the form—

$$\frac{z'-a/c}{z'-\xi}=\frac{-d/c-\xi}{z-\xi}, \quad \text{or} \quad \frac{\xi-a/c}{z'-\xi}=\frac{-d/c-\xi}{z-\xi}-1.$$

Substituting for ξ its value, this becomes

$$\frac{1}{z'-\xi}=\frac{1}{z-\xi}+\beta, \quad \text{where} \quad \beta=\frac{2c}{a+d} \qquad . \quad . \quad (15)$$

Transforming the plane in which z, z' are represented as follows

$$Z=\frac{1}{z-\xi}, \quad Z'=\frac{1}{z'-\xi} \qquad . \quad . \quad . \quad . \quad (16)$$

the transformation becomes

$$Z'=Z+\beta \qquad . \quad . \quad . \quad . \quad . \quad (17)$$

the fixed point now being at infinity. This is a translation of the plane parallel to the line $O\beta$ joining the origin to the point β.

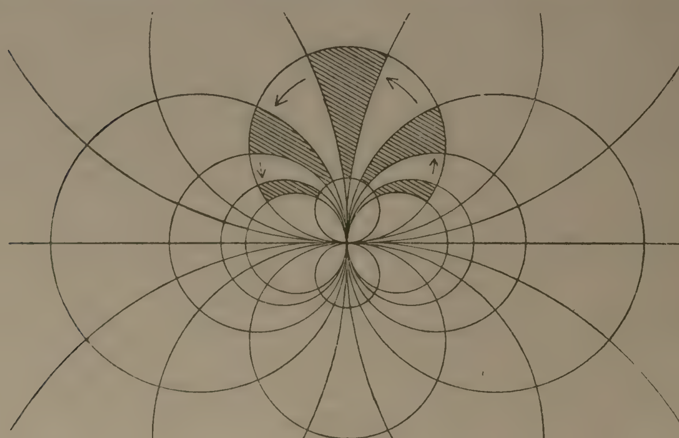

FIG. 5.

Any line parallel to $O\beta$ is transformed into itself; any other straight line is transformed into a parallel straight line. Transforming back to the zz'-plane, the point ∞ becomes the point ξ. Parallel straight lines, intersecting only at ∞, become circles intersecting only at ξ, and hence tangent at ξ. Hence, in the parabolic transformation there is a one parameter family of tangent circles through the fixed point each of which is trans-

formed into itself *; any other circle through the fixed point is transformed into a second circle tangent to it.

Fig. 5 shows a parabolic transformation in which the fixed circles are those with a horizontal tangent. The shaded regions connected by the arrows show how the parts of the plane are transformed. The circles with a vertical tangent at ξ furnish an example of a family each circle of which is transformed into some other circle of the family.

* One of the fixed circles is a straight line through ξ. The fixed line through 0, β, ∞ in the ZZ'-plane is transformed into a circle through ∞, $\xi+1/\beta$, ξ in the zz'-plane; that is, a straight line through ξ and $\xi+1/\beta$. Thus the fixed circles are those circles through ξ whose tangent is parallel to the line joining 0 and $1/\beta$.

CHAPTER II

GROUPS OF LINEAR TRANSFORMATIONS

7. Notation.—The linear transformation $z'=(az+b)/(cz+d)$ is frequently denoted by

$$\left(z,\ \frac{az+b}{cz+d}\right) \quad \text{or} \quad \left(\frac{z-\xi_1}{z-\xi_2},\ K\frac{z-\xi_1}{z-\xi_2}\right),$$

the transformation being the process of substituting the second term in the parentheses for the first. Also it will often be convenient, when not likely to lead to confusion, to represent the transformation merely by a capital letter, as S, T, U.

If a transformation S be made, and then to the transformed plane a second transformation T be applied, the single linear transformation equivalent to the succession of the two will be represented by T(S), or simply TS. ST will be different in general from TS. Thus if S is the transformation above and T is the transformation $z'=(\alpha z+\beta)/(\gamma z+\delta)$, we have, making use of Equation (5), Section 1,

$$TS=\left(z,\ \frac{(\alpha a+\beta c)z+\alpha b+\beta d}{(\gamma a+\delta c)z+\gamma b+\delta d}\right);\quad ST=\left(z,\ \frac{(a\alpha+b\gamma)z+a\beta+b\delta}{(c\alpha+d\gamma)z+c\beta+d\delta}\right);$$

which are different in general. Linear transformations are thus not commutative.

T^2, T^3, etc., are the transformations equivalent to performing T twice, thrice, etc. T^{-1} will represent the inverse of T, and $T^{-n}=(T^{-1})^n$. The identical transformation $z'=z$ is represented by 1. The fact that a transformation followed by its inverse leaves the points of the plane unchanged gives the equation $T^{-1}T=1$. Combinations of T and its inverse T^{-1} obey the ordinary laws of the addition of exponents in multiplication. In combining transformations it is easily seen that the associative law holds, *i.e.* that $S(TU)=ST(U)$. No confusion then can arise from writing STU.

To find the inverse of ST, we make on the plane transformed by ST first the transformation S^{-1}, then T^{-1}: $T^{-1}S^{-1}(ST)=T^{-1}(S^{-1}S)T=T^{-1}T=1$. Thus $T^{-1}S^{-1}$ is the transformation which, applied after ST, leaves the points unchanged; that is, $(ST)^{-1}=T^{-1}S^{-1}$. In general the inverse of ST . . . UV is $V^{-1}U^{-1} \ldots T^{-1}S^{-1}$.*

8. Definition of a Group.—*A set of transformations, finite or infinite in number, is said to form a group if*

(1) *the succession of any two transformations whatever of the set is a transformation of the set;*

(2) *the inverse of every transformation of the set is a transformation of the set.*

The definition applies to all kinds of transformations, but we shall be concerned only with groups of linear transformations. The group properties, using the notation of the last section, are these: (1) If S is a transformation of the group (not necessarily different from T), ST belongs to the group; and (2) if T is a transformation of the group, T^{-1} also belongs to the group. Since T and T^{-1} belong to the group, $T^{-1}T(=1)$ does also; hence, *every group contains the identical transformation* $z'=z$.

9. Examples of Groups.—We give here several examples of groups.

(1) *Group of Real Translations.*—All transformations of the form $z'=z+k$, where k has any real value, constitute a group. For if T is $z'=z+k_1$, T^{-1} is $z'=z-k_1$, which belongs to the set. If S is $z'=z+k_2$, ST is $z'=z+k_1+k_2$, which belongs to the set. If k be restricted to positive values, we do not have a group; for T^{-1} does not then belong to the set.

(2) *Group of Rotations about the Origin.*—The transformations $z'=e^{i\theta}z$, where θ is any real quantity, are easily shown, after the manner of the preceding case, to satisfy both group properties.

(3) *Group of the Simply Periodic Functions.*—The set $z'=z+m\omega$, where ω is constant and m is a positive or negative integer or zero, forms a group.

(4) *Group of the Doubly Periodic Functions.*—The set $z'=z+m\omega+m'\omega'$, where ω, ω' are constants and m and m' are positive or negative integers or zero, forms a group. It is assumed that the ratio ω/ω' is not real; although this is not necessary to establish the group properties.

(5) *Cyclic Group.*—The m transformations $z'=e^{2\pi in/m}z$, $n=0, 1, \ldots m-1$, form a group. They are the successive rotations about the origin through an angle $2\pi/m$.

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* The order of procedure in a combination should be carefully noted. Thus STU is U, followed by T, followed by S. If S is $z'=f_3(z)$, T is $z'=f_2(z)$, and U is $z'=f_1(z)$, the combination STU is $z'=f_3\{f_2[f_1(z)]\}$.

(6) *Group of the Anharmonic Ratios.*—The six transformations

$$z'=z,\ \frac{1}{z},\ 1-z,\ \frac{1}{1-z},\ \frac{z-1}{z},\ \frac{z}{z-1}$$

form a group. We verify by combining the transformations that both group properties are satisfied. This group is so named for the reason that if z is any one of the anharmonic ratios of four points on a line, the six anharmonic ratios are given by the transformations of the group.

10. Transforming a Group.—From any given group of linear transformations infinitely many other groups can be derived by applying a linear transformation to the plane in which z and its transformed values are represented. Let T be a transformation of the original group, and let T carry z into z'. Let a transformation G be applied to the plane, z and z' becoming z_1 and z_1' respectively. The transformation T_1 carrying z_1 to z_1' is a linear transformation equivalent to: (1) that carrying z_1 to z followed by (2) that carrying z to z' followed by (3) that carrying z' to z_1'. Hence $T_1=GTG^{-1}$.

Now, let all the transformations of the given group be transformed in this manner. We shall show that the new set of transformations form a group.

Thus, $T_1^{-1}=(GTG^{-1})^{-1}=GT^{-1}G^{-1}$, which belongs to the set since T^{-1} belongs to the original group, and the second group property is satisfied. And if $S_1=GSG^{-1}$, where S belongs to the given group, $S_1T_1=GSG^{-1}GTG^{-1}=GSTG^{-1}$, and this belongs to the set since ST is a transformation of the given group, and the first group property is satisfied. Therefore, the transformed set forms a group.

We shall find that it will frequently facilitate the study of a group to transform it in this manner. All configurations in the z_1-plane can then be carried back to the z-plane by the transformation G^{-1}.

11. Continuous and Discontinuous Groups.—Let z be an arbitrary point in the plane, and let us apply to this point all the transformations of some given group. Considering the positions of the points derived from z by these transformations, two cases will arise: either all the transformed points will lie outside a sufficiently small circle with z as its centre (excluding, of course the identical transformation which leaves the position of z unchanged); or there will be infinitely many of the trans-

formed points clustering about z. In the former case the group is said to be *discontinuous*; in the latter, *continuous*. Thus, in the group of the simply periodic functions (No. (3) of Section 9), given any finite point z, there is no transformed point within a circle of radius ω drawn about z. The group is then discontinuous. Similarly the group of the doubly periodic functions (No. (4) of Section 9) is discontinuous. In the group of real translations (No. (1) of Section 9), on the other hand, we can, by taking k sufficiently small, obtain a transformed point as near as we like to the given point. This group is continuous therefore. A group with only a finite number of transformations, as Nos. (5) and (6) of the same section, is necessarily discontinuous. We lay down the following—

DEFINITION.—*If there exist transformations of the group, the identical transformation excluded, which change the position of an arbitrary point of the* z*-plane by an arbitrarily small amount, the group is continuous. Otherwise it is discontinuous.*

If the transformed point be represented by $z+\delta z$, the transformation can be written

$$z+\delta z=(az+b)/(cz+d);$$

whence, $cz^2+(d-a)z-b+(cz+d)\delta z=0$. If the group be continuous, there will exist transformations, whatever be the point z, such that δz is arbitrarily small. Hence, there must exist transformations in which $c, d-a, b$ are arbitrarily small quantities (but not all zero, for that is but the identical transformation).

In order that the group be continuous it is not necessary that the coefficients of the transformation should vary continuously as is the case in Nos. (1) and (2). Thus consider the group $z'=e^{in\theta}z$, where n is a positive or negative integer or zero, and where θ is incommensurable with π. This is the group of rotations through positive and negative multiples of the angle θ. By rotating through the angle θ a sufficient number of times any point can be brought back as near as desired to its original position; hence, the group is continuous.

Continuous groups have been made the subject of many investigations, particularly by Lie, and have yielded important results in the theory of differential equations. However, for a reason that will appear later (Section 20, footnote), such groups

play no part in the theory to which this tract is devoted; and we shall not have occasion to consider them further.

12. The Modular Group.—In this section we make a detailed study of a well-known group. The properties of this group exemplify many theorems to be subsequently established; and the study of the group at this point furnishes a concrete background for future developments of the theory.

Consider the transformations $z'=(az+b)/(cz+d)$ in which a, b, c, d are positive or negative integers or zero subject to the condition that $ad-bc=1$. The inverse of such a transformation has the coefficients $-d, b, c, -a$ (Eqn. (3), Section 1), which are also positive or negative integers or zero and with determinant $ad-bc=1$. If $z'=(\alpha z+\beta)/(\gamma z+\delta)$ is a second transformation whose coefficients $\alpha, \beta, \gamma, \delta$ are of the desired form; then the first transformation followed by the second is a transformation with the coefficients $\alpha a+\beta c, \alpha b+\beta d, \gamma a+\delta c, \gamma b+\delta d$ (Eqn. (5), Section 1). These coefficients are integers or zero, and the determinant is $(ad-bc)(\alpha\delta-\beta\gamma)=1$. Both group properties are therefore satisfied.

All transformations $(\mathrm{z}, (\mathrm{az}+\mathrm{b})/(\mathrm{cz}+\mathrm{d}))$ *in which* a, b, c, d *are integers and* $\mathrm{ad}-\mathrm{bc}=1$ *constitute a group called the modular group.*

Let us consider the two transformations S: $(z, -1/z)$ and T: $(z, z+1)$, which belong to the modular group. It will be shown in a moment that *any transformation of the modular group can be formed by combinations of S and T with their inverses.* We verify that $S^{-1}=S$ or $S^2=1$; T^n is $(z, z+n)$; T^{-n} is $(z, z-n)$.

We shall now indicate a method of decomposing a transformation of the modular group into combinations of S and T, taking first a particular example:

$$U=(z,\ (4z+9)/(11z+25)),\quad (4\times 25)-(9\times 11)=1.$$

The method to be followed is to reduce the coefficients in the denominator by use of combinations of S and T. Since $25>11$ we replace z by $z\pm n$, n being chosen so that 25 shall be replaced by as small a number as possible. Put $z-2$ for z. We have

$$UT_1=(z,\ (4z+1)/(11z+3)),\quad \text{where}\quad T_1=(z,\ z-2)\quad \text{or}\quad T_1=T^{-2}.$$

Replacing z by $-1/z$ and clearing of fractions, the first coefficient in the denominator becomes the smaller.

$$UT_1S=(z,\ (-4/z+1)/(-11/z+3))=(z,\ (z-4)/(3z-11)).$$

Continuing,

$$\begin{aligned}
&\mathrm{UT_1ST_2}=(z,\ z/(3z+1)), \quad \text{where} \quad \mathrm{T_2}=\mathrm{T}^4=(z,\ z+4).\\
&\mathrm{UT_1ST_2S}=(z,\ -1/(z-3))\\
&\mathrm{UT_1ST_2ST_3}=(z,\ -1/z), \quad \text{where} \quad \mathrm{T_3}=\mathrm{T}^3=(z,\ z+3)\\
&\mathrm{UT_1ST_2ST_3S}=1.
\end{aligned}$$

From this last equation U is the inverse of $\mathrm{T_1ST_2ST_3S}$. Hence

$$\mathrm{U}=\mathrm{S}^{-1}\mathrm{T}_3{}^{-1}\mathrm{S}^{-1}\mathrm{T}_2{}^{-1}\mathrm{S}^{-1}\mathrm{T}_1{}^{-1}=\mathrm{ST}^{-3}\mathrm{ST}^{-4}\mathrm{ST}^2.$$

U is thus expressed in terms of S and T.

The method followed in this particular case furnishes material for a complete proof for the most general case. When $|d|>|c|$ a transformation of the form $(z, z\pm n)$ reduces the transformation to c[illegible] in which $|d|<|c|$. Putting $-1/z$ for z, we interchange c [illegible] and can reduce again. The determinant at each step is [illegible]ity, since the transformations which are combined have unit determinants. So long as $|c|\neq 1$ the substitution $(z, z\pm n)$ will not reduce d to zero, for the condition $ad-bc=1$ requires that c and d be prime to one another. We eventually reduce the denominator to unity, and $ad-bc=a=1$ and the transformation is of the form $(z, z\pm n)$, which by means of $(z, z\mp n)$ becomes equal to the identity. We then express the given transformation in terms of the transformations used in reducing it as in the example. Hence, any transformation of the modular group can be expressed in combinations of S and T.

It is clear that S and T are not the only transformations that can be regarded as generating transformations of the group. Any pair in terms of which S and T can be expressed will serve. All transformations, being expressible as combinations of S and T, will then be expressible in terms of the new pair. Thus we might have used $\mathrm{U}=(z, -1/z)$ and $\mathrm{V}=(z, -1/(z+1))$; since $\mathrm{S}=\mathrm{U}$, $\mathrm{T}=\mathrm{UV}$.

13. The Fundamental Region.—Before proceeding further with the study of the modular group it is desirable to introduce the definition of the *fundamental region*, a conception of the foremost importance in the theory of discontinuous groups.

DEFINITION.—*Two points, two curves, or two areas are said to be congruent with respect to a given group if one can be transformed into the other by some transformation of the group.*

DEFINITION.—*A region such that no two of its points are congruent with respect to a given group and such that every*

region adjacent to it contains points congruent to points in the given region is called a fundamental region for the group.

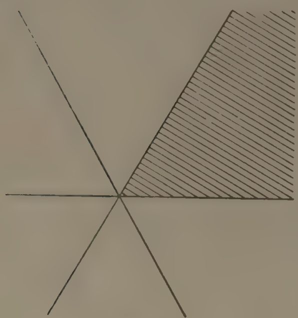

FIG. 6.

Fig. 6 shows a fundamental region for the cyclic group of six rotations $(z, e^{2ni\pi/6}z)$. No two points in the shaded region can be carried into one another by rotations through multiples of the angle $2\pi/6$; but points in any adjacent region (and in this case any other point in the plane) can be carried into some point in the region by such a rotation.

The shaded area in fig. 7 is a fundamental region, or "parallelogram of periods," for the group of the doubly periodic functions $(z, z+m\omega+m'\omega')$. The vertices of the parallelogram are the points 0, ω, ω', $\omega+\omega'$.

We note that the rays bounding the region in fig. 6 are composed of congruent points. Strictly speaking, only one of these rays should be considered as belonging to the region. A similar remark applies to fig. 7.

No point can have two congruent points in the fundamental region; for these two points, being congruent to the same point, would be congruent, which is contrary to hypothesis.

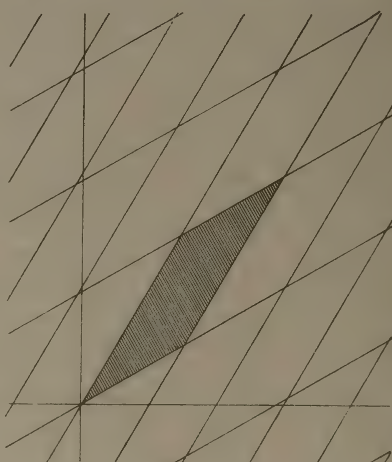

FIG. 7.

It is clear that the fundamental region is not uniquely determined. We can, for example, add to the given region a small adjacent region which has no two of its points congruent, provided we remove from the original region the part congruent to the added region. The new region is then fundamental. We note also that any region congruent to the fundamental region will also serve as a fundamental region. For example, the unshaded regions in figs. 6 and 7 are congruent to the shaded regions and any one could be employed as a fundamental region.

14. Fundamental Region for the Modular Group.—The transformation $T=(z, z+1)$ is represented geometrically by a

translation in the positive direction parallel to the real axis through a unit distance. The transformation $S = (z, -1/z)$ breaks up into the real and imaginary parts

$$x' = \frac{-x}{x^2+y^2}, \quad y' = \frac{y}{x^2+y^2}.$$

This is equivalent to

$$x' = \frac{x_0}{x_0^2+y_0^2}, \quad y' = \frac{y_0}{x_0^2+y_0^2}; \quad x_0 = -x, \quad y_0 = y;$$

that is, a reflection in the imaginary axis followed by an

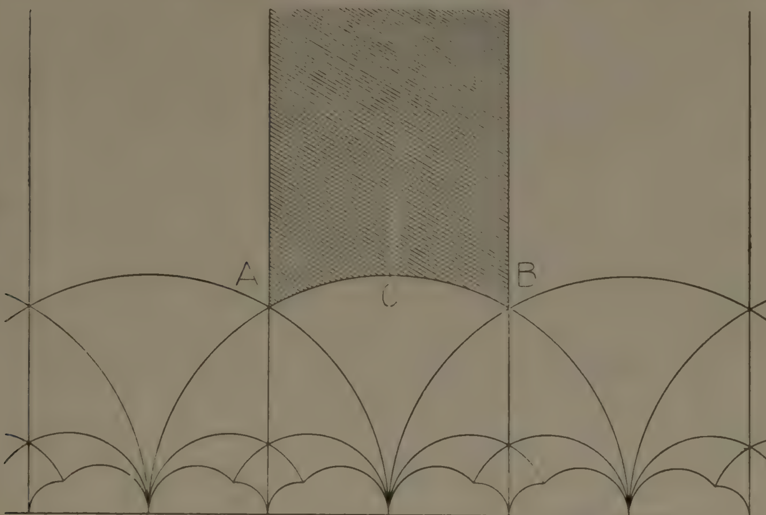

FIG. 8.

inversion in the circle whose centre is at the origin and whose radius is unity.

Consider the region lying outside the unit circle $x^2+y^2=1$ and between the lines $x = \pm\frac{1}{2}$. This is the shaded region in fig. 8. The translation T carries this region into a congruent one abutting along the boundary $x = \frac{1}{2}$. The translation T^{-1} carries it into a congruent region abutting along the boundary $x = -\frac{1}{2}$. The transformation S carries it into a congruent region abutting along the circular portion of the boundary. Hence the given region satisfies the second part of the definition of a fundamental region.

It remains to show that there are no two points of the fundamental region congruent. To do this it will be proved that the transform of a point of the region lies outside the region by every transformation of the group except the identical transformation. It is necessary to consider three cases, according to the value of c in the transformation $(z, (az+b)/(cz+d))$:—

(1) $c=0$. The determinant $ad=1$ gives $a=d=\pm 1$.

The transformation is then a translation $(z, z+n)$ where n is a positive or negative integer. This translation carries a point of the region into a position outside the region.

If $c \neq 0$, we can write the transformation

$$z' - a/c = -1/c(cz+d).$$

(2) $c=\pm 1$. $|z' \mp a| = 1/|z \pm d|$.

Now $|z \pm d|$ is the distance from z to the point $\mp d$, and $|z' \mp a|$ is the distance from z' to $\pm a$, a and d being integers. The point z in the given region lies outside the unit circles drawn about the integral points on the real axis. Then, $|z \pm d| > 1$. Hence $|z' \mp a| < 1$, and z' lies inside the unit circle about $\pm a$. It is thus outside the given region.

(3) $|c| \geqq 2$. $|z' - a/c| = 1/c^2 |z + d/c|$.

The points a/c and d/c lie on the real axis. The points of the given region lie above the line $y=\frac{1}{2}$, then $|z+d/c|>\frac{1}{2}$. $|z' - a/c| < 2/c^2 \leqq \frac{1}{2}$, since $c^2 \geqq 4$. The point z' is below $y=\frac{1}{2}$, and is therefore outside the given region.

Hence when z lies in the given region no transformation of the group carries it into some other point of the region. The given region then contains no two congruent points and is therefore a fundamental region. Since we have proved that the modular group has a fundamental region, we have established the fact that it is a discontinuous group.

We found that the transformations, T, T^{-1}, S, carry the fundamental region into congruent regions adjacent to it. The transformation carrying the fundamental region into an adjacent region R carries the three regions surrounding it into three regions surrounding R. There are transformations of the group carrying the fundamental region into these regions adjoining R.

There are transformations of the group carrying the fundamental region into regions adjacent to the new regions, and so on. Continuing this process, we cover the whole upper half-plane with regions congruent to the fundamental region. This partition of the plane is shown in fig. 8.

We get a similar division of the lower half-plane by a reflection in the real axis. The fundamental region and its reflected region are inverses in the real axis; and by Theorem 10(b) the various transforms of the region in the lower half-plane will be the reflection of the corresponding regions in the upper half-plane.

Infinitely many regions cluster along the real axis. The points on this axis are for this reason called the *points of discontinuity* of the group.

15. Fuchsian Groups, or Groups with Principal Circle.— The coefficients of the modular group being real, any real point is transformed into a real point by a transformation of the group. The real axis is then a *fixed circle* for each transformation of the group. This group is a member of an important class of groups having the property designated and called Fuchsian groups.*

DEFINITIONS.—*If all the transformations of a discontinuous group leave one and the same circle invariant, and if points in the interior of the fixed circle are transformed into points in the interior by each of the transformations, the group is called a Fuchsian group.*

The fixed circle is called the principal circle of the group.

A reference to the article on Types of Linear Transformations and to figs. 3, 4, and 5, where the fixed circles are drawn, shows that the transformations of a Fuchsian group are:—

(1) hyperbolic transformations with the fixed points on the principal circle;

(2) elliptic transformations with the fixed points inverse to one another in the principal circle;

(3) parabolic transformations with the fixed point on the principal circle.

* This term was introduced by Poincaré, *Comptes rendus*, xcii. (1881), p. 333. It is not very appropriate, but will be retained here.

No loxodromic transformation can belong to the group; for the only loxodromic transformations possessing fixed circles transform the interior of each of the circles into its exterior.

It will simplify matters to transform a given Fuchsian group, in the manner described in Section 10, by a transformation which carries the fixed circle into the real axis. Any Fuchsian group can be thus transformed into one with the real axis as principal circle, the interior and exterior of the fixed circle becoming the two halves of the plane separated by the real axis. The real axis being the principal circle, any real point is transformed into a real point. If $z'=(az+b)/(cz+d)$ is a transformation of the group, a, b, c, d are real quantities, or at most contain a common complex factor which can be removed by division.

The requirement that the interior of the fixed circle should be transformed into itself (and, of course, the exterior into itself also) gives a further condition on the constants. Breaking the transformation into its real and imaginary parts:

$$x'=\frac{ac(x^2+y^2)+(ad+bc)x+bd}{(cx+d)^2+c^2y^2}, \quad y'=\frac{(ad-bc)y}{(cx+d)^2+c^2y^2},$$

we see that the necessary and sufficient condition that y' have the same sign as y, and hence z' and z be on the same side of the real axis, is that $ad-bc>0$. Dividing numerator and denominator by the *real* quantity $\sqrt{(ad-bc)}$, the determinant becomes $+1$, the coefficients remaining real.*

16. Fundamental Region for a Fuchsian Group.—Consider a Fuchsian group with the real axis as principal circle. Let a be a point of the plane at which there are no infinitesimal transformations. Since the group is assumed to be discontinuous, such a point exists. We shall suppose a to be in the upper half-plane. None of the points congruent to a lies in its neighbourhood. There will exist a small region R about a all of whose points are transformed into points outside R by the transforma-

* It was pointed out in the early pages of the book that the determinant can always be considered $+1$. If, however, $ad-bc<0$, the determinant becomes $+1$ only on division by $\sqrt{(ad-bc)}$, an imaginary quantity, and the coefficients no longer remain real.

It is easily seen that the real transformations with negative determinants are the loxodromic transformations of angle π mentioned in Section 6 (c), which carry the interior of each fixed circle into its exterior.

tions of the group, the identical transformation excluded. That is, if R is sufficiently small, no two of its points are congruent. Now let R be enlarged by the addition of adjacent regions so long as this can be done without adding points congruent to each other or to points already in the region. When R is enlarged as much as possible in this way, we have a fundamental region for the group; for no two points of the region are congruent, and no adjacent region can be added without introducing points congruent to points already in the region.

Let b be a point on the boundary of this region (see fig. 9). There is a transformation of the group carrying b and a portion

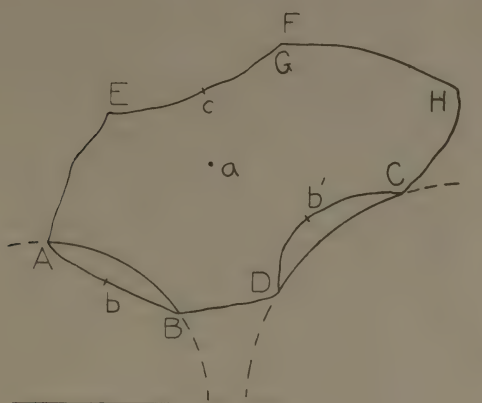

FIG. 9.

of the boundary on one or both sides of b into some boundary point b' and a portion of the boundary near b'. If this were not the case, we could add to the given region a small adjacent region at b without introducing points congruent to any in the fundamental region—which is contrary to hypothesis. Call this transformation S. Let AB be the part of the boundary containing b which is transformed by S into CD containing b'. *The fundamental region is carried by S into an adjacent region, CD being the common boundary.* Similarly S^{-1} carries the fundamental region into an adjacent region, AB being the common boundary.

Considering a point c not on AB or CD, we get a portion EF of the boundary containing c as interior or end point congruent

to another portion HG by a transformation T of the group. Further, AB, CD, EF, HG contain no parts in common; for a point lying just outside the region near a common portion of the boundary would have two congruent points inside the region, which is impossible. Continuing the above process, we conclude that *the fundamental region is bounded by a finite or infinite number of pairs of congruent curves.*

Almost without exception the Fuchsian groups that have been investigated are those in which the fundamental region is bounded by a *finite* number of pairs of congruent sides; and it is with this case only that we shall be concerned.

The fundamental region may lie entirely above the real axis, as in the case of the modular group; or it may lie on both sides of the real axis and contain a portion of the axis. In the latter case we need consider only that portion lying above the axis, for the remaining portion can be derived from this by a reflection. But it should be borne in mind that both portions belong to the fundamental region.

Concerning the nature of the boundary of the above fundamental region we know nothing. We propose now to simplify the region by replacing the boundary by another whose nature we know. Let a circle with its centre on the x-axis be drawn through A and B (fig. 9). This circle is perpendicular to the axis. It is transformed by S into the circle through C and D with its centre on the x-axis. [The circle through A, B is orthogonal to the x-axis. Since angles are preserved and the x-axis is transformed into itself, this circle is transformed into a circle orthogonal to the x-axis which we know passes through C, D.] We can replace the former boundaries AbB and Cb'D by these two congruent circles; for any portion of the original region removed near one boundary is replaced by a congruent region near the other boundary. The modified region is a fundamental region. Proceeding in a similar manner for the other parts of the boundary, we get the entire boundary composed of arcs of circles. Hence, *the fundamental region can be chosen as a circular polygon, the bounding circles having their centres on the* x-*axis and being arranged in congruent pairs.*

We shall speak of the transformations of the group by which the congruent pairs of sides are connected as the *generating transformations of the group.* It will be proved later that any

transformation of the group can be expressed as a combination of these generating transformations.

17. The Cycles.—The sides of the fundamental region being arranged in congruent pairs, to each point on the boundary there corresponds in general one other point congruent to it and lying on the boundary. This is not generally true, however, of the

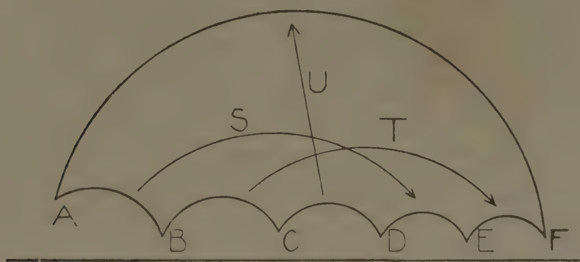

FIG. 10.

vertices of the polygon, each of which lies on two sides. Corresponding to a vertex there may be one congruent vertex, there may be several, or there may be none.

DEFINITION.—*Each set of congruent vertices of the fundamental polygon is called a cycle.*

A cycle may consist of one vertex or of several.

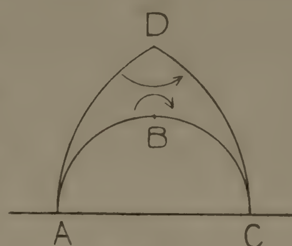

FIG. 11.

In fig. 10 is shown a fundamental region in which the congruent sides are connected by arrows, S, T, U being the generating transformations. Performing the transformation S, AB is transformed into ED, E being the transform of A. Performing T^{-1}, EF goes into CB, C being the transform of E. Performing U, CD goes into AF, A being the transform of C. The vertices A, E, C are congruent vertices, and constitute a cycle. Starting with B and making in succession T, U^{-1}, S^{-1}, we get the vertices F, D, B. Hence B, F, D form a second cycle.

In fig. 11 there are three cycles: AC, B, and D. A vertex lying on the axis as A is called a *parabolic point.* The two circles there are tangent. Since the points congruent to a point on the

axis also lie on the axis the vertices of a cycle containing a parabolic point are all parabolic points. One of the vertices of the fundamental region for the modular group is a parabolic point—the vertex at infinity in the shaded region of fig. 8. It constitutes a cycle.

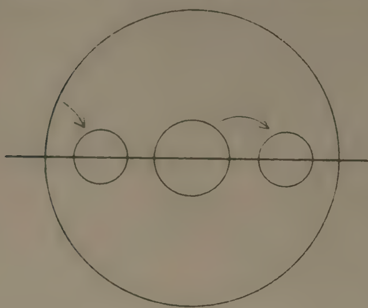

FIG. 12.

Fig. 12 is an example of a fundamental region lying on both sides of the principal circle. The region considered is that lying within the large circle and without the three small circles. It is the fundamental region for Weber's group mentioned later. This particular region has no vertices.*

Two useful theorems connected with the cycle will now be established—

THEOREM 13.—*The sum of the angles in each cycle not containing parabolic points is a submultiple of* 2π.

Let $A_1, A_2, \ldots, A_n$ be the vertices of the cycle. Let T_1 be the transformation by which A_1 is transformed into A_2, a side adjacent to A_1 becoming a side adjacent to A_2; T_2 the transformation carrying A_2 and an adjacent side into A_3 and an adjacent side; and so on to T_n, which carries A_n and an adjacent side into A_1 and an adjacent side. Each of the transformations T_i is one of the generating transformations or its inverse. Consider the n regions congruent to the fundamental region by applying to the fundamental region the n transformations of the group: T_n, T_nT_{n-1}, . . ., $T_n \ldots T_3T_2$, $T_n \ldots T_2T_1$. Call the fundamental region R_0 and the congruent regions just mentioned $R_1, R_2, \ldots, R_n$ respectively. T_n carries A_n and an adjacent side into A_1 and an adjacent side. Hence R_1 has a vertex at A_1 and adjoins R_0 along one of the sides terminating at A_1. Since angles are preserved, R_1 has a vertex of angle A_n †

* The treatment here departs from that of Poincaré in the papers in the early volumes of the *Acta Mathematica*, where the idea of the cycle first appeared. There the region considered would be only that part above the axis, the four segments of the axis forming part of the boundary. The region then has eight vertices and four cycles. The cycles of this kind are "open cycles" in Poincaré's terminology.

† No confusion will arise by representing by A_i the angle at the vertex A_i.

at A_1. The transformation T_nT_{n-1} carries A_{n-1} to A_1. Further, R_2 adjoins R_1 along a side terminating at A_1; for T_{n-1} carries R_0 into a region R' adjacent to R_0 at A_{n-1}, and the transformation T_n carries R_0 into R_1 and R' into R_2 adjacent to it. Continuing, we get the regions $R_3, \ldots, R_{n-1}, R_n$ as shown in the figure, the vertices at A_1 having the angles $A_{n-2}, \ldots, A_2, A_1$. If the angle at the point A_1 is not yet filled, we continue with the regions $R_{n+1}, R_{n+2}, \ldots$ derived from R_0 by the transformations $T_n \ldots T_1T_n$, $T_n \ldots T_1T_nT_{n-1}$, etc. Now, none of the angles $A_1, \ldots, A_n$ is zero, for the bounding sides are circles with their centres on the x-axis and no two such circles can be tangent at a point not lying on the axis. Hence, by the above process the angle about the point is eventually filled. Now, none of the angles $A_2, \ldots, A_n$ can overlap the original angle A_1; for the points in these angles are congruent to points in the angles $A_2 \ldots, A_n$ of the original region R_0, and any points common to the angle A_1 and one of these angles would have another congruent point in the fundamental region, which is impossible. Hence the angle A_1 of $R_n, R_{2n}, \ldots$, or R_{pn} coincides with the angle A_1 of R_0. If R_n, each of the angles appears once, and the sum is 2π; if R_{2n}, each appears twice, and the sum is $2\pi/2$; in general each appears p times, where p is some integer, and the sum of the angles is $2\pi/p$, which was to be proved.

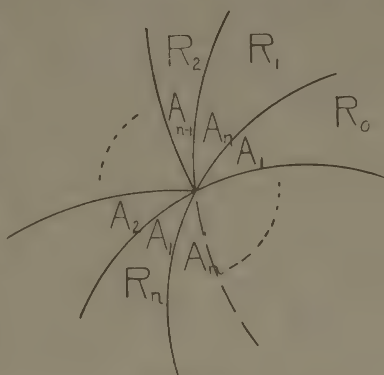

FIG. 13.

THEOREM 14.—*To each cycle not containing parabolic points there corresponds a relation between the generating transformations of the group.*

We have just found that when the transformation $(T_n \ldots T_2T_1)^p$ is applied to R_0, the transformed region R_{pn} has the angle A_1 in its original position. The points in this angle must coincide with their original positions, since otherwise the angle A_1 would contain pairs of congruent points—which is impossible since A_1 belongs to the fundamental region. But if three or more points are unchanged, all points are unchanged, and the transformation

is the identity. Hence $(T_n T_{n-1} \ldots T_2 T_1)^p = 1$, a relation between the generating transformations of the group.

These two theorems can be illustrated by the modular group. The point C (fig. 8) constitutes a cycle of angle $2\pi/2$. The sides at C are connected by the transformation S; whence the second theorem gives $S^2 = 1$. A and B make a cycle of angle $2\pi/3$. A returns to itself by the transformation T followed by S; whence $(ST)^3 = STSTST = 1$. This can be verified directly from the equations of the transformations.

18. The Transforms of the Fundamental Region.—Let $T_1, T_2, \ldots, T_n$ be the generating transformations connecting the n pairs of congruent sides of the fundamental region, R_0. As has been pointed out, R_0 can be transformed into a region adjacent to it along any one of its sides by means of one of the transformations $T_1, \ldots, T_n, T_1^{-1}, \ldots T_n^{-1}$. Let T' be some transformation of the group, and let R' be the region into which it transforms R_0. Let $A'B'$ be any side of the polygon R'. *There is a transformation of the group transforming R_0 into a region R'' adjoining R' along the arc $A'B'$.* For let AB be the side of the polygon R_0 which is carried by the transformation T' into $A'B'$. Let T_k be the generating transformation carrying R_0 into R_k adjacent to R_0 along the side AB. Then T' carries R_k into a region R'' adjacent to R' along the side $A'B'$. Therefore $T'T_k$, which is a transformation of the group, carries R_0 into R''.

Starting with the region R_0, we transform it by means of the generating transformations and their inverses into regions adjoining it along its $2n$ sides. We can then transform R_0 into regions adjoining these new regions along each of their sides, and so on. Unless the group consists of only a finite number of transformations we get in this way infinitely many transforms of the fundamental region. The question now arises: How much of the plane is covered by these repetitions of the fundamental region?

THEOREM 15.—*The transforms of the fundamental region in the above manner (or, if the region lies on both sides of the x-axis, of that part lying above the axis) cover the entire upper half-plane.*

To prove this, it will be shown that any given point Q in the

upper half-plane can be reached by repetitions of the fundamental region according to the method indicated in a finite number of steps. Let P be a point in the fundamental region, and join P, Q by a straight line. PQ lies entirely above the real axis; let K be its minimum distance from the axis. We shall suppose for the moment that the fundamental region does not extend to infinity. Let M be its maximum distance from the x-axis. Let AB be the side of the fundamental region from which the line PQ issues (fig. 14). Construct the region R_1 adjacent to R_0 along the side AB. Let PQ leave R_1 by the side A_1B_1. Construct R_2, the transform of R_0 adjoining R_1 along the side A_1B_1, etc.* Consider C_iD_i, the portion of the line PQ lying in the region

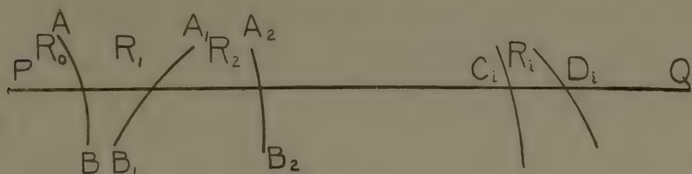

FIG. 14.

R_i. Let $z'=(a_iz+b_i)/(c_iz+d_i)$ be the transformation by which R_i is derived from R_0. Then C_iD_i is congruent to an arc CD joining two sides of R_0. The length of C_iD_i is $\int_{C_i}^{D_i}|dz'|$. By Theorem 12, if $a_id_i-b_ic_i=1$, this length is $\int_C^D\frac{|dz|}{|c_iz+d_i|^2}$. Now, $|c_iz+d_i|^2=(c_ix+d_i)^2+c_i^2y^2$; and from the equations of Section 15 $\frac{1}{|c_iz+d_i|^2}=\frac{y'}{y}>\frac{K}{M}$. Therefore $C_iD_i>\frac{K}{M}\int_C^D|dz|=\frac{K}{M}\times$ length of arc CD. Suppose there are infinitely many segments C_iD_i. Their sum is greater than K/M times the sum of the lengths of infinitely many arcs CD crossing R_0 from side to side. Now this latter sum is infinite. For the only case in which it would not be infinite is that in which all the arcs CD, save for a finite number, cross the region near the vertices of the polygon; but this would mean that PQ winds round a point at which vertices meet, which is impossible. The assumption that the number of segments C_iD_i is infinite leads then to the conclusion that the

* In case PQ crosses a vertex of one of the polygons a slight deformation of the line will make it cross a side.

sum of their lengths is infinite. But this is impossible since the sum of their lengths is not greater than PQ, a finite quantity. Hence Q can be reached by repetitions of the fundamental region in a finite number of steps.

This proof does not hold if the fundamental region extends to infinity, for in that case we cannot conclude that y, the imaginary coordinate of a point on CD, is bounded. In that case consider the congruent regions adjacent to R_0. One or more of these adjacent regions will not extend to infinity save in the one case in which the fundamental region consists of only two sides with infinity as a parabolic point. This is the case in which the fundamental region is that between two vertical lines, the "period strip" of the simply periodic functions, and the transformations are real translations; in this case the half-plane is obviously covered. We lay this case aside. Now, take P in this new region; y is bounded, and the preceding proof applies. Since any point Q in the upper half-plane can be reached in a finite number of steps, it follows that the whole upper half-plane is covered by these repetitions of the fundamental region.

We note that all the transformations involved in these repetitions of the fundamental region have been combinations of $T_1, \ldots, T_n$ with their inverses. The group has no other transformations. For let T be any transformation of the group. It carries R_0 into a region R in the upper half-plane. R must coincide with one of the regions R_k already found; for if there were overlapping, the region common to R and R_k would contain points having two congruent points in R_0, which is impossible. Whence $T = T_k$, where T_k is the transformation carrying R_0 into R_k and is a combination of the generating transformations. Hence—

THEOREM.—*If $T_1, \ldots T_n$ are the transformations connecting the* n *pairs of congruent sides of the fundamental region of a Fuchsian group, any transformation of the group can be expressed as a combination of $T_1, \ldots, T_n$ with their inverses.*

If the group has only a finite number of transformations, the half-plane will be covered by a finite number of repetitions of the fundamental region. If the group has an infinite number of transformations, the half-plane will be divided into an infinite number of regions. Since any finite point Q above the axis can be reached in a finite number of steps, any finite region lying

entirely above the axis will contain in its interior only a finite number of regions or parts of regions. Hence the points about which infinitely many regions cluster lie on the real axis. These cluster points may comprise all the points of the axis, as in the case of the modular group; or only certain points of the axis, as in the case of groups in which the fundamental region crosses the axis.

The division of the lower half-plane into regions can be made by reflecting the upper half-plane in the real axis. If R_0 lies entirely above the axis, its reflection will be a fundamental region R_0' lying entirely below the axis. The repetitions of R_0' will, by virtue of Theorem 10 (*b*), be the reflections of the repetitions of R_0. If the fundamental region crosses the axis, the lower half of the region can be chosen as the reflection of the upper half; and the reflection of the whole half-plane supplies the lower parts of all the transformed regions.

19. Groups without Principal Circles.—A discontinuous group which possesses no principal circle (called *Kleinian* by Poincaré) exhibits several of the properties already found for Fuchsian groups. For example, the fundamental region can be simplified by the use of circular arc boundaries, although now the bounding circles will not be orthogonal to a fixed circle. The sum of the angles of a cycle not composed of parabolic points (*i.e.* vertices where the adjacent sides are tangent) will be a submultiple of 2π; and each such cycle yields a relation between the generating transformations of the group. And we can partition the area surrounding the fundamental region by means of transforms of the region, as in Section 18. It is here that we come to an essential difference, however—the copies of the fundamental region do not fill the interior of a circle. In the simpler cases the whole plane may be covered by a finite number of regions (as shown in fig. 23), or by an infinite number clustering about a point (as in fig. 7), but in general the division is much more complicated. A common phenomenon is that in which only a portion of the plane, bounded by an irregular curve, is covered; and several different fundamental regions are necessary to obtain the division of the whole plane (just as in the Fuchsian case two regions, one on each side of the principal circle, are usually required).

Poincaré has introduced a method of treating these groups by means of three dimensional transformations. Each transformation is equivalent to a certain even number of inversions in circles in the plane. The transformations in the plane will be the same if these inversions be made in spheres with the same centres and radii as the circles; and furthermore by the use of these spheres the transformations are defined for points above and below the complex plane. It can then be shown that there exists a fundamental region for these generalised transformations bounded by spherical surfaces orthogonal to the complex plane. The parts of the plane contained in this three-dimensional fundamental region furnish the several fundamental regions in the plane whose repetitions cover the whole plane.*

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* Poincaré, *Mémoire sur les groups kleinéens*, Acta Math., vol. iii. (1883), pp. 49–92; Forsyth, *Theory of Functions*, pp. 610–618.

CHAPTER III

AUTOMORPHIC FUNCTIONS

20. Automorphic Functions.—Automorphic functions are the natural generalisation of the circular, hyperbolic, elliptic, and certain other functions of elementary analysis. A circular function, as $\sin z$, has the property that it is unchanged in value if z is replaced by $z+2m\pi$, where m is any integer; that is, the function is unchanged in value if z be subjected to any transformation of the group $(z, z+2m\pi)$. A hyperbolic function, as $\sinh z$, is unchanged in value if z be subjected to a transformation of the group $(z, z+2m\pi i)$; an elliptic function, as the Weierstrassian function $\wp(z)$, retains its value under transformations of a group of the form $(z, z+m\omega+m'\omega')$.

DEFINITION.—*A function* $F(z)$ *is said to be automorphic with respect to a discontinuous group if* $F\left(\frac{az+b}{cz+d}\right)\equiv F(z)$, *where* $\left(z, \frac{az+b}{cz+d}\right)$ *is any transformation of the group.**

It will be proved later that there are automorphic functions connected with any discontinuous group. For the present we assume this fact and proceed to derive some of the properties of the functions. We shall confine our attention to groups (Fuchsian or Kleinian) for which the fundamental region has a finite number of sides, and to those automorphic functions having no other singularities than poles in the fundamental region. To avoid long circumlocutions in the statement of the

* There are no functions, other than constants, which are unchanged in value under all the transformations of a *continuous* group. For, let z_0 be a point at which such a function has no singularity. Since the group has infinitesimal substitutions, there are infinitely many points in the neighbourhood of z_0 at which $F(z)=F(z_0)$. It is a well-known theorem that a function which has the same value at infinitely many points in the neighbourhood of a point at which it has no singularity is identically equal to that value. Hence, $F(z)\equiv F(z_0)$, a constant. It is for this reason that we confined our attention in the study of groups solely to those which are discontinuous.

theorems we shall call such functions *simple automorphic functions*.* Owing to its automorphic character we shall have a complete knowledge of the function when we know its behaviour in the fundamental region.

21. Properties of Simple Automorphic Functions.—

THEOREM A.—*A simple automorphic function takes on its value* p *times, or some multiple thereof, at a vertex belonging to a cycle the sum of whose angles is* $2\pi/$p.

$F(z)$ is said to take on its value s times at a point z_0 if it can be written $F(z)=F(z_0)+(z-z_0)^s\phi(z)$, where $\phi(z_0)\neq 0$.† In considering the number of points at which the function has the value $F(z_0)$ the point z_0 is counted s times. Let the function be simply automorphic, and let A be a vertex belonging to a cycle of angle $2\pi/p$. We can write $F(z)=F(A)+(z-A)^s\phi(z)$, where $\phi(A)\neq 0$, and we are to prove that s is a multiple of p.

It was found in Section 17 that in the Fuchsian case there is a transformation T of the group with A as fixed point which rotates points near A through an angle $2\pi/p$. The other fixed point is A′, the reflection of A in the real axis. T then is the transformation $\dfrac{z'-A}{z'-A'}=e^{2\pi i/p}\dfrac{z-A}{z-A'}$. A similar fact can be shown for the Kleinian group, A′ being a point not in the neighbourhood of A. Applying this transformation to the function, we get

$$F(z')=F(A)+(z'-A)^s\phi(z')=F(A)+(z-A)^s e^{2\pi is/p}\left[\phi(z')\left(\frac{z'-A'}{z-A'}\right)^s\right].$$

The function, being automorphic, is unchanged in value. By taking z near A, z' is also near A; and, since A′ is a distant point, the function in square brackets can be made to differ arbitrarily little from $\phi(z)$. Hence $e^{2\pi is/p}=1$, or s is a multiple of p; which establishes the theorem.

If there are n vertices in the cycle, np regions have a vertex at A. Dividing the s-fold value there equally among them, s/np will be said to belong to the fundamental region. A similar remark applies to each of the other congruent

* If the group is Fuchsian, the simple automorphic function is usually called a Fuchsian function.

† In the case of a pole at z_0, $F(z_0)$ is absent and s is negative in the formula.

vertices, yielding s/p (an integer) points belonging to the fundamental region at which the value F(A) is taken on.

In the neighbourhood of a parabolic point P the behaviour is less simple. Since infinitely many regions cluster about P the function has there an essential singularity (*i.e.* a singularity which is not a mere pole, for near a pole F(z) can take on no value more than a finite number of times). Let F(P) be the value approached by F(z) when z approaches P in the fundamental region. We can show that there is a parabolic transformation of the form $\frac{1}{z'-\mathrm{P}}=\frac{1}{z-\mathrm{P}}+\beta$ by which the function is unchanged. It is unchanged by any repetition of this transformation: $\frac{1}{z'-\mathrm{P}}=\frac{1}{z-\mathrm{P}}+n\beta$. By transforming the point P to infinity by the transformation $\mathrm{Z}=1/(z-\mathrm{P})$ the transformations become $\mathrm{Z}'=\mathrm{Z}+n\beta$, a group of simply periodic transformations. From this we can show that F is a function of t having at most poles where $t=e^{2\pi i\mathrm{Z}/\beta}$. Hence—

THEOREM B.—*At a parabolic point a simple automorphic function is a function of* t *having at most poles, where* $\mathrm{t}=\mathrm{e}^{2\pi i/(z-P)\beta}$.

This kind of singularity is called *logarithmic*. The number of times that F(z) takes on its value at the vertices of the cycle to which P belongs is determined by the index m in the expression for F in the form $\mathrm{F}(z)=\mathrm{F}(\mathrm{P})+t^m\phi(t)$.

THEOREM C.—*A simple automorphic function, not identically zero, has as many zeros as it has poles in the fundamental region.*

If $f(z)$ has no other singularities than poles in a region enclosed by a contour and if the function and its derivative are continuous on the contour and the function does not vanish there, then the integral $\frac{1}{2\pi i}\int d\log f(z)$ taken around the contour in a counter-clockwise direction is equal to N – M, where N is the number of zeros and M the number of poles in the region.* We shall apply this theorem to the automorphic function F(z), integrating around the boundary of the fundamental region. We suppose first that the function has no zeros nor poles on the boundary. Consider the parts of the

* Whittaker and Watson's *Modern Analysis*, p. 119; or Osgood, *Lehrbuch der Funktionentheorie*, 2nd ed., p. 333.

integral arising from two congruent sides AB, CD (fig. 15), $\int_D^C d\log F(z) + \int_A^B d\log F(z)$. Since $F(z)$ has the same values along the congruent sides, this sum is $\int_B^A d\log F(z) + \int_A^B d\log F(z) = 0$. Similarly the parts arising from other pairs of congruent sides cancel; and $N - M = 0$ or $N = M$, which was to be proved.

The cases in which $F(z)$ has zeros or poles on the boundary require further consideration. If there is a zero or a pole on the side AB, there is one also on the side CD. Change the path of integration by means of two congruent arcs as shown in fig. 16, and the integrals cancel as before. Only one of the points is included in the new region: and it is clear that

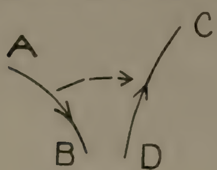

FIG. 15.

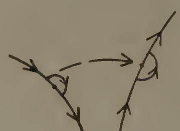
FIG. 16.

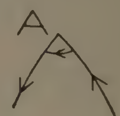

FIG. 17.

since the points are congruent only one should be considered as belonging to the region.

If $F(z)$ has a zero or pole at a vertex A, we integrate as in fig. 17. We shall suppose it is a zero of order s. Let $2\pi/p$ be the sum of the angles in the cycle to which A belongs. The integral $\frac{1}{2\pi i}\int d\log F(z)$ taken in a counter-clockwise direction entirely around A is s; hence, the integral taken in the clockwise direction along the small arc shown in the figure is $-s \cdot \angle A/2\pi$. There will be a similar integral at each of the congruent vertices, and these will not cancel. Hence $N - M = -(s/2\pi)\Sigma \angle A_i = -s/p$; or $N + s/p = M$. But s/p is precisely the number of zeros belonging to the region which have not been included in the contour; and we see that the theorem holds here also. The case of a pole at A is treated similarly.

We can show likewise that the case of a pole or a zero at a parabolic point produces no exception.

The methods above dispose of every case except those in which there are infinitely many zeros or poles on the boundary of the region. But the function cannot have infinitely many

poles without having an essential singularity which we assume it does not have; and it cannot have infinitely many zeros without being identically zero. Hence the theorem is established.

THEOREM D.—*A simple automorphic function which has no singularities in the fundamental region is a constant.*

Let $F(z)$ be such a simple automorphic function; and let its value at z_0, a point in the fundamental region, be C. $F(z)-C$ is a simple automorphic function with a zero at z_0 and having no poles. Such a function, according to Theorem C, is identically zero. Therefore, $F(z)-C\equiv 0$, or $F(z)\equiv C$.

THEOREM E.—*A simple automorphic function which is not a constant takes on every value in the fundamental region the same number of times.*

Consider the function $F(z)-C$, where C is a constant. This is a simple automorphic function with the same poles as $F(z)$. It has as many zeros as it has poles. The zeros of $F(z)-C$ are the points at which $F(z)$ takes on the value C. Since C may be any constant, we conclude that the number of times $F(z)$ takes on any given value is equal to the number of its poles, which establishes the theorem.

THEOREM F.—*Between two simple automorphic functions belonging to the same group there exists an algebraic relation.*

Let $F_1(z)$ and $F_2(z)$ be the two functions with k_1 and k_2 poles respectively. We are to show that there exists a relation

$$\Phi(F_1, F_2) = A_1F_1^mF_2^n + A_2F_1^mF_2^{n-1} + \ldots + A_{(m+1)(n+1)} = 0,$$

where A_1, A_2, etc., are constants not all zero, the relation being independent of z. The function Φ, whatever values be given the constants, is a simple automorphic function which has no poles other than at the points where occur the poles of $F_1(z)$ and $F_2(z)$. The number of its poles is not greater than mk_1+nk_2. We can choose the constants $A_1, \ldots, A_{(m+1)(n+1)}$ so that Φ will have zeros at $(m+1)(n+1)-1$, points c_1, c_2, etc., not coinciding with the poles of $F_1(z)$ or $F_2(z)$ by satisfying the $(m+1)(n+1)-1$ equations

$$A_1F_1^m(c_i)F_2^n(c_i) + \ldots + A_{(m+1)(n+1)} = 0, \quad i = 1, 2, \ldots, (m+1)(n+1)-1.$$

Constants, A_1, A_2, etc., not all zero, can always be chosen to satisfy these equations since there is one more constant than equations to be satisfied. Now if m and n be large enough $(m+1)(n+1)-1 > mk_1+nk_2$, and the Fuchsian function Φ has

more zeros than possible poles. According to Theorem C, $\Phi(F_1, F_2)\equiv 0$, the poles having mutually cancelled. Hence the theorem.

It will generally happen, if the algebraic relation be found in this manner, that $\Phi(F_1, F_2)$ is reducible: $\Phi(F_1, F_2) = \Phi_1(F_1, F_2)\Phi_2(F_1, F_2) \ . \ . \ . \ \Phi_n(F_1, F_2)$. Some one at least of the irreducible factors must vanish, say $\Phi_1(F_1, F_2)\equiv 0$. This irreducible relation will contain both functions unless one of them is a constant, in which case the irreducible relation is of the form $F_1(z)-c\equiv 0$.

It is easy to see in the general case what the degree of this irreducible equation in F_1 and in F_2 will be. The degree in F_1 is the number of values of F_1 satisfying the equation when F_2 is given a fixed value. For a fixed value of F_2 there are k_2 values of z at which $F_2(z)$ takes on that value, k_2 as before being the number of poles of $F_2(z)$. For each of these values $F_1(z)$ has a value satisfying the equation. Hence, $\Phi_1(F_1, F_2)=0$ is of degree k_2 in F_1. Similarly it is of degree k_1 in F_2. For particular functions some of the values of F_1 arising from the k_2 values of z may always be equal (as in the functions F and F^2), in which case the degree in F_1 is less than k_2.

THEOREM G.—*Any simple automorphic function can be expressed rationally in terms of two simple automorphic functions connected with the group.*

Let us consider two functions $F_1(z)$ and $F_2(z)$ with k_1 and k_2 poles respectively whose irreducible algebraic relation $\Phi(F_1, F_2)=0$ is of degree k_2 in F_1 and k_1 in F_2, as mentioned in the preceding paragraph. Let $F_3(z)$ be a third Fuchsian function connected with the group. Each pair of values of F_1, F_2 satisfying the algebraic equation is given by a *single* point z in the fundamental region (save for a few exceptional points). This value of z gives one value only of $F_3(z)$. F_3 is then a single-valued function of F_1 and F_2 so long as these satisfy the algebraic relation $\Phi(F_1, F_2)=0$. Such a function, when it has only polar singularities, as in the case of F_3, we know from the theory of algebraic functions to be a rational function of the variables in the algebraic equation. Hence—

$$F_3 = \frac{A_1 F_1^m F_2^n + \ . \ . \ .}{B_1 F_1^p F_2^q + \ . \ . \ .} = R(F_1, F_2),$$

which was to be established.

The derivative of an automorphic function is not, in general, automorphic. Let $z'=(az+b)/(cz+d)$ be a transformation of the group, where $ad-bc=1$. Then $F(z')=F(z)$; but

$$F'(z')=\frac{dF(z')}{dz'}=\frac{dF(z)}{dz}\frac{dz}{dz'}=F'(z)(cz+d)^2,$$

and $F'(z')\neq F(z)$, except in the case $(cz+d)^2=1$ for all transformations of the group. This condition holds only in case the transformations are all translations, $z'=z+k$.* We note that the ratio of the derivatives of two automorphic functions is automorphic; *i.e.* $F_1'(z')/F_2'(z')=F_1(z)/F_2(z)$.

THEOREM H.—*Two simple automorphic functions connected with a group satisfy, in general, an algebraic differential equation of the first order.*

For $\dfrac{dF_1}{dz}\Big/\dfrac{dF_2}{dz}$, or $\dfrac{dF_1}{dF_2}$, is a Fuchsian function. By the preceding theorem this is expressible rationally in general in terms of F_1 and F_2. Hence, $dF_1/dF_2=R(F_1, F_2)$, the variable z not explicitly appearing.

For convenience let us represent an automorphic function by $x(z)$. Concerning the inverse function $z(x)$ we shall now prove the following remarkable theorem—

THEOREM I.—*If* x(z) *is an automorphic function of* z, z *can be expressed as a function of* x *by the quotient of two solutions of a linear differential equation of the second order of the form*

$$\frac{d^2y}{dx^2}=R(w, x)y, \quad \textit{where} \quad \Phi(w, x)=0,$$

R being rational in w *and* x *and* Φ *a polynominal.*

Consider the function $y_1(z)=\sqrt{(dx/dz)}$. We shall show first that $\dfrac{1}{y_1}\dfrac{d^2y_1}{dx^2}$ is an automorphic function. Let us subject z to a transformation of the group, and represent by y_1' and $x'(=x)$ the transformed values of the functions. Making use of the equations $dx'/dz'=(cz+d)^2dx/dz$ and $1/y_1^2=dz/dx$, we obtain the following:—

* The singly and doubly periodic functions constitute this exceptional class in which the derivative is also automorphic.

$$y_1' = \sqrt{\frac{dx'}{dz'}} = (cz+d)\sqrt{\frac{dx}{dz}} = (cz+d)y_1,$$

$$\frac{dy_1'}{dx'} = \frac{dy_1'}{dx} = (cz+d)\frac{dy_1}{dx} + cy_1\frac{dz}{dx} = (cz+d)\frac{dy_1}{dx} + \frac{c}{y_1},$$

$$\frac{d^2y_1'}{dx'^2} = (cz+d)\frac{d^2y_1}{dx^2} + c\frac{dz}{dx}\frac{dy_1}{dx} - \frac{c}{y_1^2}\frac{dy_1}{dx} = (cz+d)\frac{d^2y_1}{dx}.$$

Hence,

$$\frac{1}{y_1'}\frac{d^2y_1'}{dx'^2} = \frac{1}{y_1}\frac{d^2y_1}{dx^2}.$$

This function being automorphic, is expressible rationally in terms of x and some other automorphic function w, and w and x are joined by an algebraic relation $\Phi(w, x)=0$. Whence y_1 satisfies an equation

$$\frac{d^2y}{dx^2} = \mathrm{R}(w, x)y, \quad \text{where} \quad \Phi(w, x)=0.$$

Now let y_2 be a second solution of this equation independent of the first; y_1 and y_2 satisfy an equation of the form

$$y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = c,$$

where c is a constant different from 0. [In general the expression equals $ce^{-\int \mathrm{P}dx}$, where $\mathrm{P}(x)$ is the coefficient of dy/dx. In this case $\mathrm{P}(x)=0$.*] Dividing by y_1^2,

$$\frac{1}{y_1}\frac{dy_2}{dx} - \frac{y_2}{y_1^2}\frac{dy_1}{dx} = \frac{c}{y_1^2} = c\frac{dz}{dx}.$$

Whence, integrating,

$$\frac{y_2}{y_1} + k = cz$$

or

$$z = \frac{y_2/c + ky_1/c}{y_1} = \frac{y_3}{y_1},$$

where y_3 is a solution of the equation. The theorem is thus established.

We note that two independent solutions of the equation are $y = \surd(dx/dz)$ and $y = z\surd(dx/dz)$.

22. The Thetafuchsian Series.—We have heretofore assumed the existence of functions automorphic with respect to a given

* Forsyth, *A Treatise on Differential Equations*, 4th ed., p. 114.

group. We shall now demonstrate their existence, and in fact construct such functions, by means of a series discovered by Poincaré.* We treat here the case of Fuchsian groups.

Let the unit circle with centre at the origin be the principal circle of the group; and let

$$z_i = \frac{a_i z + b_i}{c_i z + d_i} \quad i = 0, 1, 2, 3, \ldots, \quad a_i d_i - b_i c_i = 1$$

be the transformations belonging to the group. We shall suppose the identical transformation to be $z_0 = z$. Let $\mathrm{H}(z)$ be a rational function of z none of whose poles lie on the principal circle. We consider now the following series

$$\Theta(z) = \sum_i (c_i z + d_i)^{-2m} \mathrm{H}\left(\frac{a_i z + b_i}{c_i z + d_i}\right).$$

We shall prove presently that this series is convergent (save for certain exceptional points) both within and without the principal circle, provided m is an integer greater than 1. This series is called the *Thetafuchsian series.* Let us assume the convergence of the series and investigate the properties of the function $\Theta(z)$ which it defines.

If z be subjected to a transformation of the group $z_j = (a_j z + b_j)/(c_j z + d_j)$, the series becomes

$$\Theta(z_j) = \sum_i \left(c_i \frac{a_j z + b_j}{c_j z + d_j} + d_i\right)^{-2m} \mathrm{H}\left(\frac{(a_i a_j + b_i c_j)z + a_i b_j + b_i d_j}{(c_i a_j + d_i c_j)z + c_i b_j + d_i d_j}\right).$$

If we represent by z_{ij} the transform of the point z when the j-transformation followed by the i-transformation is made, the second factor of each term of the series is $\mathrm{H}\left(\dfrac{a_{ij} z + b_{ij}}{c_{ij} z + d_{ij}}\right)$. The first factor is $\left(\dfrac{(c_i a_j + d_i c_j)z + c_i b_j + d_i d_j}{c_j z + d_j}\right)^{-2m}$ or $(c_j z + d_j)^{2m}(c_{ij} z + d_{ij})^{-2m}$. The first of these factors, being independent of i, is the same for all terms. The series then is

$$\Theta(z_j) = (c_j z + d_j)^{2m} \sum_i (c_{ij} z + d_{ij})^{-2m} \mathrm{H}\left(\frac{a_{ij} z + b_{ij}}{c_{ij} z + d_{ij}}\right).$$

But the expression under the sign of summation is $\Theta(z)$; for the i-transformations of the original series included all the trans-

* *Comptes rendus*, xcii. (1881), p. 333; *Mémoire sur les fonctions fuchsiennes*, Acta Mathematica, vol. i. (1883), p. 193.

formations of the group, and the ij-transformations of the last series include also all the transformations of the group, the order of the terms being changed. Hence,

$$\Theta(z_j)=(c_jz+d_j)^{2m}\Theta(z).$$

By means of Thetafuchsian series we can set up infinitely many automorphic functions. Let $\Theta_1(z)$ and $\Theta_2(z)$ be two series formed from two rational functions $H_1(z)$ and $H_2(z)$, the integer m being the same. Consider the function $F(z)=\Theta_1(z)/\Theta_2(z)$.

$$F(z_j)=\frac{(c_jz+d_j)^{2m}\Theta_1(z)}{(c_jz+d_j)^{2m}\Theta_2(z)}=F(z).$$

$F(z)$ is then an automorphic function.

It is easy to form other combinations of Θ-series which are automorphic. Thus if the numerator be a sum of terms of the form $\Theta_1\Theta_2 \ . \ . \ . \ \Theta_k$ where the integers used in forming the series are $m_1, m_2, \ . \ . \ . \ m_k$ respectively, and the denominator be also a sum of such terms, the function will be automorphic if $m_1+m_2+ \ . \ . \ . \ +m_k$ is the same for all the terms. If $F(z)$ is an automorphic function, we found that its derivative is multiplied by $(c_jz+d_j)^2$ on performing a transformation of the group. Hence $\Theta(z)/[F'(z)]^m$ is an automorphic function, and we can readily deduce other combinations which are automorphic.

23. Convergence of the Series.—There are certain points of the plane at which individual terms of the series become infinite. If a is a pole of $H(z)$, there is an infinite term when $(a_iz+b_i)/(c_iz+d_i)=a$. That is, a term becomes infinite whenever z approaches any point congruent to a pole of $H(z)$. A term becomes infinite also when $z=-d_i/c_i$; that is, at points congruent to the point infinity. Since each transformation carries the exterior of the circle into itself, these latter points are all exterior to the circle. Let a small circle be drawn about each pole of $H(z)$ and let the regions within these circles and all the circular regions congruent to them be excluded from consideration. Similarly let a large circle K be drawn with no point congruent to infinity on its exterior, and let the region outside this circle and the congruent circular regions about the points $-d_i/c_i$ be excluded. The portion of the plane that remains—that inside K and outside the circles about the other excepted

points—is finite in area. Let C_0 be any contour lying within this region and containing none of the excepted regions on its interior. We suppose also that no two points within C_0 are congruent. C_0 may lie entirely within the principal circle, or entirely outside it, or it may contain a portion of the circle on its interior in case the fundamental region of the group contains a portion of the circle. We shall now prove that within C_0 the Thetafuchsian series converges uniformly.

To do this we shall prove that whatever value z has within C_0 the absolute value of the term $(c_iz+d_i)^{-2m}H(z_i)$ is less than a constant U_i, and the series $U_1+U_2+\ldots$ converges. Let M_i and m_i be the maximum and minimum values of $|c_iz+d_i|^{-1}$ in C_0. Now $|c_iz+d_i|^{-1}=|c_i|^{-1}|z+d_i/c_i|^{-1}$, and $|z+d_i/c_i|$ is the distance from the point z to the excepted point $-d_i/c_i$. This excepted point lies within the circle K and without the region C_0. If D be the greatest distance from a point of C_0 to a point on K, and d be the least distance from a point of C_0 to any of the circles surrounding the excepted points, we have $M_i<1/c_iD$ and $m_i>1/c_id$; whence $M_i<dm_i/D$.

Let $C_1, C_2, \ldots$ be the regions congruent to C_0; and let the areas of the regions be $A_0, A_1, A_2. \ldots$ Applying the formula found for areas in Section 5, we get the following—

$$A_i=\iint_{C_0}|c_iz+d_i|^{-4}dxdy>\iint_{C_0}m_i^4dxdy=m_i^4A_0.$$

Hence $m_i^4<A_i/A_0$, and $M_i^4<d^4A_i/D^4A_0$. Now, $M_0^4+M_1^4+\ldots<(d^4/D^4A_0)(A_0+A_1+\ldots)=S$, a finite quantity, since the sum of the areas $A_0, A_1, \ldots$ is less than the area of K. Hence $M_i<S^{\frac{1}{4}}$, whatever the value of i; and $M_i^{2m}=M_i^{2m-4}M_i^4<S^{(m-2)/2}d^4A_i/D^4A_0$. This inequality is true on the assumption that $m\geqq 2$.

Consider now $H(z_i)$. If z is in C_0 each transformed point z_i lies outside the excepted regions around the poles of $H(z)$. Hence $H(z_i)$ is finite and we can choose a constant H greater than the value of the function for every point outside the excepted regions.

We have now the material for establishing the convergence of the series. Each term $(c_iz+d_i)^{-2m}H(z_i)$ is less in absolute value, whatever the value of z within C_0, than the constant $M_i^{2m}H$. But

$$\Sigma M_i^{2m}H<(S^{(m-2)/2}Hd^4/D^4A_0)(A_0+A_1+\ldots)=S^{m/2}H.$$

The series $\Sigma M_i H$ thus converges; hence the original series converges absolutely and uniformly in C_0; and the sum $\Theta(z)$ is an analytic function of z.*

At one of the excepted points a term of the series becomes infinite, but if this term be suppressed the series converges. $\Theta(z)$ then has a pole at the excepted point. Let us consider the function in a fundamental region within the principal circle. There is a point in the fundamental region congruent to each pole of $H(z)$ within the principal circle. Hence, *the number of poles of* $\Theta(z)$ *within the fundamental region is in general equal to the number of poles of* $H(z)$ *within the principal circle.* In particular cases two poles of $H(z)$ may be congruent points, and the order of the pole at their common congruent point in the fundamental region is not greater than the highest order of the two poles.

In a fundamental region outside the principal circle the number of poles of $\Theta(z)$ is equal to the number of poles of $H(z)$ outside the region increased by $2m$, the latter arising from a term $(c_i z+d_i)^{-2m}$ at a point congruent to infinity. In a group having a fundamental region crossing the circle the number of poles is the sum of the two preceding numbers. The function has essential singularities at the points of the principal circle where the excepted points cluster.

If $H(z)$ has no poles within the principal circle, or if the poles are at congruent points and are of such a nature that the poles of the infinite terms arising from them cancel, $\Theta(z)$ has no singularities within the circle. Poincaré has shown, in the memoir cited above, that of all such functions, formed with a given value of m, only a finite number are linearly independent [*i.e.* for a group whose fundamental region does not cross the principal circle].

The poles of the automorphic function $F(z)=\Theta_1(z)/\Theta_2(z)$ arise from the poles of the numerator and the zeros of the denominator; its zeros arise from the zeros of the numerator and the poles of the denominator. Now, the function $F(z)$ has the same number of poles as of zeros in the fundamental region. We conclude from this that for all functions $\Theta(z)$ formed with a given m

* The convergence of a similarly formed series for a group without a principal circle can be established by slight and obvious changes of the proof just given, provided infinity is not a point of discontinuity of the group. The series is called Thetakleinian.

the difference between the number of zeros and of poles is a constant.*

It should be noted that in case all points of the principal circle are points of discontinuity of the given group—that is, points at which infinitely many fundamental regions cluster—every point of the circle is an essential singularity for the functions defined by the series outside the circle and the function inside. Each of the functions then has the circle as a natural boundary. Since the function inside cannot be continued analytically into the one outside, the two functions, although defined by the same series, are distinct functions.

* Poincaré proved that in the case of a fundamental region lying entirely within the principal circle, the number of zeros exceeds the number of poles by $m(n-1-S/2\pi)$, where $2n$ is the number of sides of the fundamental region and S is the sum of the angles of its vertices.

CHAPTER IV

THE RIEMANN-SCHWARZ TRIANGLE FUNCTIONS *

24. The Differential Equation.—Given a linear differential equation of the second order

$$\frac{d^2y}{dx^2}+\mathrm{P}(x)\frac{dy}{dx}+\mathrm{Q}(x)y=0$$

and two independent particular solutions $y=y_1(x)$ and $y=y_2(x)$; then the general solution is of the form $y=\mathrm{A}y_1+\mathrm{B}y_2$. The connection with linear transformations, which we have already noted in Section 21, Theorem I, arises in the following manner. Let $z=y_1/y_2$ be the quotient of the two solutions; and let $z'=y_1'/y_2'$ be the quotient of two other solutions, $y_1'=ay_1+by_2$, $y_2'=cy_1+dy_2$. Then

$$z'=\frac{ay_1+by_2}{cy_1+dy_2}=\frac{az+b}{cz+d}.$$

In particular y_1' and y_2' may be other branches of y_1 and y_2 in case these solutions are not single-valued. The determinant of this linear relation will vanish only if y_1' and y_2' differ by a constant factor.

We shall consider the hypergeometric equation with three singularities and with the real exponent differences λ, μ, ν. The equation can in all cases be reduced to the following—

$$x(x-1)\frac{d^2y}{dx^2}+\{(2-\lambda-\mu)x+\lambda-1\}\frac{dy}{dx}+\frac{(1-\lambda-\mu)^2-\nu^2}{4}y=0,$$

an equation whose singularities are at the points 0, 1, ∞.

By the ordinary method of the substitution of a power series in the differential equation and determining the coefficients, we find that if a is a point not coinciding with 0, 1, or ∞, there are two independent solutions of the form—

* Riemann's *Vorlesungen über die hyperg. Reihe* in his *Werke, Nachträge*; Schwarz, *Journal für Math.*, **75** (1873), p. 292.

$$y_{1,a}=(x-a)\{1+P_1(x-a)\}$$
$$y_{2,a}=1+P_2(x-a)$$

where P_1 and P_2 are ascending power series in $x-a$ having no constant terms. Similarly at the singular points we find the solutions

$$y_{1,0}=x^\lambda\{1+P_3(x)\}$$
$$y_{2,0}=1+P_4(x)$$
$$y_{1,1}=x^\mu\{1+P_5(x-1)\}$$
$$y_{2,1}=1+P_6(x-1)$$
$$y_{1,\infty}=\left(\frac{1}{x}\right)^{\frac{1}{2}(1-\lambda-\mu+\nu)}\left\{1+P_7\left(\frac{1}{x}\right)\right\}$$
$$y_{2,\infty}=\left(\frac{1}{x}\right)^{\frac{1}{2}(1-\lambda-\mu-\nu)}\left\{1+P_8\left(\frac{1}{x}\right)\right\}.$$

Further, the coefficients in the series P_3, . . ., P_8 are real, and the coefficients in P_1 and P_2 are real if a is real.

25. Conformal Map of Half-Plane.—Let $z=y_1(x)/y_2(x)$ be the ratio of two linearly independent solutions. z is in general a many-valued function of x, but in a region not surrounding the singular points any branch of the function will be single-valued. Such a region is the upper half-plane, and we now propose to find into what region in the z-plane the upper half-plane is mapped by the function $z=y_1/y_2$.

Consider a point a on the real axis, between 0 and 1, say. The function $z_a=y_{1,a}/y_{2,a}=(x-a)\{1+P(x-a)\}$ maps the neighbourhood of the point $x=a$ on the neighbourhood of $z_a=0$. The coefficients of the series $P(x-a)$ are real; hence, the real axis near a is transformed into the real axis in the z_a-plane near the origin. By virtue of a remark in the preceding section, $z=(a_1z_a+b_1)/(c_1z_a+d_1)$, a linear transformation by which the real axis in the z_a-plane becomes a circle in the z-plane. Hence, the real axis near a is mapped on an arc of a circle in the z-plane. If the series for z_a does not converge throughout all the region $0-1$, we consider z_b, the ratio of the principal solutions at a point b, such that the region of convergence of the series for z_b overlaps that of the preceding series. We get the axis at b mapped on an arc in the z-plane. It is a continuation of the arc previously found, since it coincides with it in the map of the common region of convergence. Continuing, we get the line $0-1$ becoming, by the transformation $z=y_1/y_2$, a circular arc in the z-plane.

At the origin the ratio of the two solutions of the preceding section is $z_0 = y_{1,0}/y_{2,0} = x^\lambda\{1+P_0(x)\}$. Taking the real branch of the function, when $x>0$, $z_0>0$; the positive part of the real axis near $x=0$ becoming the positive part at $z_0=0$. But when x passes above the origin and then traces the negative portion of the axis, z_0 is in general no longer real. Although $P_0(x)$ remains real, containing no fractional powers, x^λ does not. Writing $x=\rho e^{i\theta}$, the circuit of the origin gives $x=\rho e^{i\pi}$ on the negative part of the axis. Whence $x^\lambda = \rho^\lambda e^{i\lambda\pi}$, and z_0 has the factor $e^{i\lambda\pi}$. The axis $x<0$ becomes a line in the z-plane making an angle $\lambda\pi$ with the positive real axis, the region in the upper half-plane at $x=0$ being transformed into the region inside this angle. By means of $z=(a_0z_0+b_0)/(c_0z_0+d_0)$ this line becomes an arc in the z-plane making with the arc previously found an angle $\lambda\pi$.

Similarly we can show that the part of the real axis for $x>1$ is transformed into an arc in the z-plane meeting the transform of the axis $0<x<1$ at an angle $\mu\pi$. If we continue the mapping of the parts $x>1$ and $x<0$ to infinity, a consideration of the solutions $y_{1,\infty}$, $y_{2,\infty}$ shows that the two arcs meet at an angle $\nu\pi$. Thus we have the result that—

The ratio of any two solutions $z=y_1(x)/y_2(x)$ *maps the upper half-plane of the variable* x *on a circular arc triangle in the plane of* z, *the sides of the triangle being the transforms of the lines* 01, 1∞, *and* $\infty 0$. *Its angles are* $\lambda\pi$, $\mu\pi$, $\nu\pi$.

Let us denote the vertices of this triangle by z_0, z_1, z_∞. If, starting at a, we had traversed the real axis passing below the singular points we should have had a map of the lower half of the plane on a triangle $z_0z_1z'_\infty$ with angles $\lambda\pi$, $\mu\pi$, $\nu\pi$, the side z_0z_1 being common to the two triangles. It is easy to find the relation between these triangles. Since each of the auxiliary functions introduced, z_a, z_0, etc., are series with real coefficients, conjugate imaginary points in the x-plane become conjugate imaginary points in the z_a-plane, etc. These go by the linear transformation into pairs of inverse points with respect to the circle into which the real axis is transformed in the z-plane. Hence, *the map of the lower half-plane just found is derived from the triangle* $z_0z_1z_\infty$ *by an inversion in the side* z_0z_1.

Now, z is a many-valued function of x. If we start with a point on the negative part of the axis, using the branch of the

function which mapped the line 0∞ on $z_0 z_\infty$, and map the lower half-plane with this branch, passing below the singularities, we get a triangle which is an inversion of the original triangle in the side $z_0 z_\infty$. In general, there is a branch of the function mapping either half-plane on a triangle derived from the map of the other half-plane by inversion in a side. These triangles are in general infinite in number and overlap.

26. The Triangle Function.—The relation $z = y_1(x)/y_2(x)$ defines x as a function of z, say $x = f(z)$. Under what circumstances is $f(z)$ a single-valued function? The triangles furnish a ready answer. It is that there must be no overlapping; for a point z belonging to two maps of the x-plane gives two different values of the function. At z_0 the two triangles mapping the upper and lower half-plane form an angle $2\lambda\pi$, and in order that there shall be no overlapping by continued reflections in the sides meeting at z_0, it is necessary and sufficient that $2\lambda\pi$ be a submultiple of 2π. A similar remark applies to the other vertices. Hence, *the angles of the triangle* $\lambda\pi$, $\mu\pi$, $\nu\pi$ *are submultiples of* π, *or* λ, μ, ν *are the reciprocals of integers.**

For convenience let each triangle which is a map of the upper half-plane be shaded. The shaded and unshaded triangles alternate. Each shaded (unshaded) region is derived from an adjacent unshaded (shaded) region by inversion. Hence, any shaded (unshaded) region is derived from any other shaded (unshaded) region by an even number of inversions; that is, by a linear transformation. It is easy to see that *these transformations form a group* (both group properties of Section 8 being satisfied); and that *a shaded and an adjacent unshaded triangle together form a fundamental region for the group.*

The fundamental region is a map of the whole x-plane. By a transformation of the group this region is carried into some other map of the plane, congruent points being the transforms of the same point in the x-plane. Hence, if z and z' are any two congruent points $f(z') = f(z)$; or f(z) *is automorphic with respect*

* It is possible also to have zero values. If $\lambda = 0$, in finding $y_{1,0}$ and $y_{2,0}$ the method of substituting a series of undetermined coefficients yields but one solution. A second solution with a logarithmic term can be found, and we can show that in the triangle $z_0 z_1 z_\infty$ the sides $z_0 z_1$ and $z_0 z_\infty$ are tangent at z_0. The angle of the triangle is thus $\lambda\pi$ as before.

to the group. This automorphic function is called a *triangle function.*

27. The Three Kinds of Groups.—The groups arising in connection with the triangle functions exhibit strikingly different characteristics according as the sum of the angles of the triangle is equal to, less than, or greater than π. We divide the groups, for purposes of study, then into the three cases:—

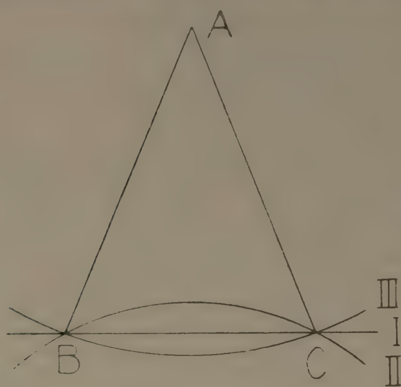

FIG. 18.

Case I. $\lambda+\mu+\nu=1$.

Case II. $\lambda+\mu+\nu<1$.

Case III. $\lambda+\mu+\nu>1$.

It will be a simplification to transform the z-plane as follows. Let the sides of the triangle meeting at, say, z_0 meet again at z'. Let a linear transformation be made carrying z' to infinity. Two of the sides of the triangle then become straight lines. In fig. 18 is shown the character of the third side in each of the three cases.

28. Case I. $\lambda+\mu+\nu=1$.—In this case the third side is also a straight line. It is obvious that by continued reflections in the sides *the triangles are infinite in number and cover the whole plane.* There are four possible cases: in which λ, μ, ν have the values (i.) $\frac{1}{2}, \frac{1}{2}, 0$; (ii.) $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$; (iii.) $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$; and (iv.) $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. The division of the plane for (ii.) is shown in the accompanying figure.* On trans-

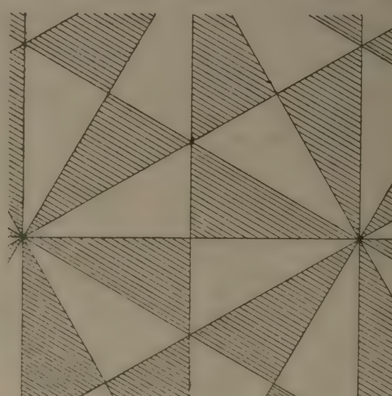

FIG. 19.

* It will be observed that by combining several of the triangles in this case we can get a parallelogram whose opposite sides are congruent by transformations of the group. Thus in fig. 19 the twelve triangles having either a side or a vertex

forming back to the original z-plane we get infinitely many triangles with the point z', which in this case is the common intersection of the three sides, as a cluster point.

29. Case II. $\lambda+\mu+\nu<1$.—The third side BC (fig. 18) is convex toward A, hence A lies outside the circle of which BC is

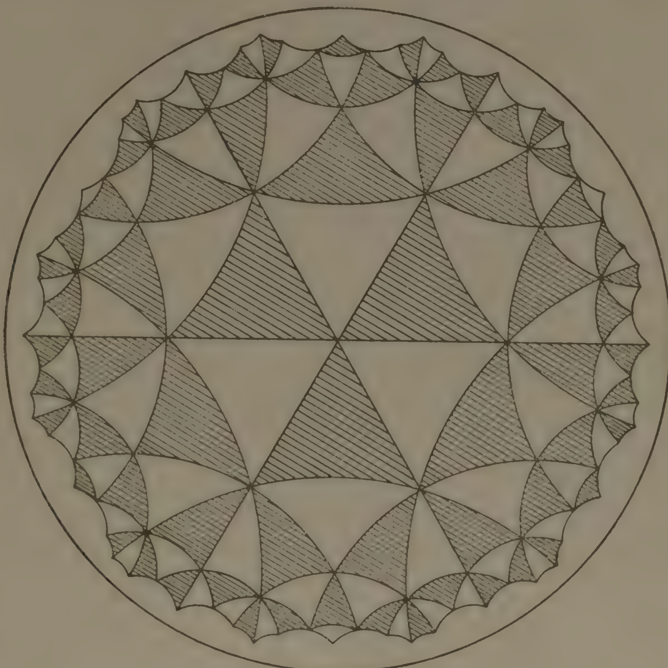

FIG. 20.

an arc. From A we can draw a real tangent AT to this circle. With A as centre and AT as radius (T being the point of tangency) let a circle K be constructed. K is orthogonal to each of the sides of the triangle. By an inversion in any side

on the vertical line through the centre of the figure is such a parallelogram, the translations which carry a side into an opposite side transforming the system of triangles into itself. This is then the period parallelogram for a doubly periodic group, which is a subgroup of the original group.

The reduction of the triangle functions to elliptic integrals in the four cases in which $\lambda+\mu+\nu=1$ has been effected by A. Polakov, *Recueil Math. de Moscou*, **27** (1911), p. 424.

of the triangle, K is transformed into itself. The new triangles arising will also have their sides orthogonal to K, and for all succeeding inversions K will be a fixed circle. The linear transformations arising from an even number of these inversions have K as a principal circle. *The group is Fuchsian.* It follows from Theorem 15, Section 18, that *the triangles will be infinite in number and entirely fill the circle K.*

There are infinitely many possible values for λ, μ, ν such that their sum is less than unity. Fig. 20 shows the case in which the values are $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{4}$. The modular group is an example of the case of the values $\frac{1}{2}$, $\frac{1}{3}$, 0. The fundamental region shaded in fig. 8 is composed of two triangles. The single triangle is the half of the shaded region formed by a vertical line through the point C. The same figure can be used to illustrate the case of the values $\frac{1}{3}$, $\frac{1}{3}$, 0. In this case the shaded region is a single triangle and the fundamental region consists of the shaded region and an adjacent region. The group is no longer the modular group, being composed of only a part of the transformations of that group; that is, it is a subgroup of the modular group.

30. Stereographic Projection.—Before discussing the third case we shall introduce a widely used method of representing the z-plane on the surface of a sphere. Let T be the z-plane, and let P be a point not lying in the plane. Now let an inversion be made in a sphere K with its centre at the point P. T is transformed into a sphere S passing through P (fig. 21). The same inversion transforms the sphere S into T. This inversion is called the *stereographic projection* of the sphere on the plane or of the plane on the sphere. Obviously the correspondence of the points of S and T is one-to-one.

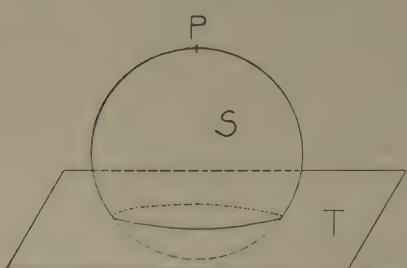

FIG. 21.

The two outstanding properties of inversion in a sphere are (1) that spheres (including planes) are transformed into spheres; and (2) that the angles between lines are preserved in magnitude. From these we can state the following facts concerning corre-

sponding figures on S and T: (1) angles are preserved in magnitude; (2) a circle in T is transformed into a circle on S, and conversely; (3) points of T inverse with respect to some sphere R will correspond to points on S inverse with respect to the transformed sphere R′, and conversely.

By virtue of Thereom 7, Section 3, we can state at once that *every one-to-one and directly conformal transformation of the sphere into itself corresponds to a linear transformation in the plane.* Thus the rotation of the sphere about an axis corresponds to a transformation of the plane whose fixed points are the stereographic projections of the extremities of the axis.*

It is easy to find the transformation of the sphere into itself corresponding to an inversion of the plane in a circle C. The effect on the plane is the same if the inversion be made in a sphere R with the same centre and radius as C. This sphere is orthogonal to the plane. By means of (3) above we see that the corresponding transformation of the sphere is an inversion in R′, the transform of R by means of the inversion of the stereographic projection. R′ is orthogonal to S and intersects it in C′, the circle corresponding to C. Hence *inversions of S in orthogonal spheres correspond to ordinary inversions in the plane T.* Such an inversion changes the sign of the angles on S, but an even number of the inversions preserves the signs of the angles and corresponds to a linear transformation of the plane.

A particularly useful inversion in an orthogonal sphere is a reflection in a diametral plane. The succession of two such reflections (or in general of an even number) is equivalent to a rotation of the sphere, the line of intersection of the two diametral planes being the axis of rotation. Further, reflection in a diametral plane does not change the magnitudes of figures on the sphere, and transforms great circles into great circles.

31. Case III. $\lambda+\mu+\nu>1$.—There are four possible cases, λ, μ, ν having the values (i.) $\frac{1}{2}$, $\frac{1}{2}$, $1/n$, where n is any integer;

* This is the most general transformation of the sphere into itself by *rigid motion.* For, given any rigid motion, there is one fixed point A on the sphere, since every linear transformation has a fixed point. The centre O is also fixed in rigid motion; hence the line OA is a fixed axis, and the motion is a rotation. The circles on the sphere in planes perpendicular to OA are the fixed circles. Projecting these on the plane, we note that the transformation corresponding to a rotation of the sphere is elliptic in type.

(ii.) $\frac{1}{2}, \frac{1}{3}, \frac{1}{3}$; (iii.) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$; and (iv.) $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$. In these cases the vertex A of the triangle (see fig. 22) lies within the circle of which BC is an arc; there is no real tangent to the circle from A, and no common orthogonal circle. It will now be shown that ABC can be projected stereographically on a sphere in such a manner that the three sides become arcs of great circles.

Through A draw the chord TT′ of the circle of the third side which is bisected at A (TT′ is perpendicular to the radius through A). The circle K, with A as centre and AT as radius, is intersected by each side of the triangle at two points which are at opposite ends of a diameter. Now let the plane be projected stereographically on a sphere S with A as centre and AT as radius. The points on the circle K are unchanged in position. The three sides of the triangle become three circular arcs each of which passes through opposite ends of a diameter. That is, the sides of the corresponding triangle A′B′C′ on the sphere are arcs of great circles.

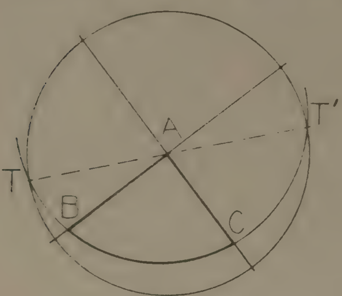

FIG. 22.

From the remarks of the last paragraph of the preceding section we see that inversions in the sides of the triangle ABC correspond to reflections in the diametral planes of the sides of the triangle A′B′C′. The sides of the new triangles arising by these reflections are also arcs of great circles; and the continued inversions in the sides of succeeding triangles in the plane are all reflections in the diametral planes of the corresponding sides of triangles on the sphere. The group of linear transformations arising from combinations of even numbers of the inversions in the plane is *a group of rotations of the sphere.*

The area of the triangle A′B′C′ is unchanged by reflection in a diametral plane. Hence by repeated reflections the whole surface of the sphere will be covered by a finite number of triangles. Projecting these triangles on the plane, we can state that *in the case* $\lambda+\mu+\nu>1$ *the triangles are finite in number and fill the whole plane.*

We can determine the number of the triangles in any case. The area of the spherical triangle A′B′C′ is $\pi(\mathrm{AT})^2(\lambda+\mu+\nu-1)$,

and the area of the whole spherical surface is $4\pi(AT)^2$. Hence the number of triangles is $N = 4/(\lambda + \mu + \nu - 1)$. In the four possible cases mentioned above we have (i.) $N = 4n$; (ii.) $N = 24$; (iii.) $N = 48$; and (iv.) $N = 120$. In each case half will be shaded and half unshaded. The number of copies of the fundamental region, which consists of some two adjacent triangles, is $N/2$; from which we conclude that the group of linear transformations has $N/2$ distinct transformations, including the identical transformation.

32. The Regular Solids.—Case III., in which the triangles in the plane can be projected stereographically into ordinary spherical triangles all equal in size, is connected with the groups of rotations about axes and reflections in planes of symmetry by which a regular solid is transformed into itself.*

We treat here only case (iii.), in which λ, μ, ν have the values $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. The division of the plane into the forty-eight triangles is shown in fig. 23. If these be projected stereographically on a sphere S of which K is a great circle, we get a system of forty-eight spherical triangles. These are shown in fig. 24. The triangles are not equilateral.

Now consider the division of the plane into eight triangles indicated by heavy lines in fig. 23. Each of these larger triangles is composed of six of the smaller ones. The angles of each are $\pi/2$, $\pi/2$, $\pi/2$; and each of these triangles becomes an equilateral triangle—an octant—on the sphere. Now let the bounding arcs of these triangles be replaced by the chords joining their extremities, and we have the edges of a regular octahedron inscribed in the sphere. This octahedron is shown in fig. 24.†

Now let us consider the rotations of the sphere which correspond to the transformations in the plane by which a shaded triangle, say, is carried into another shaded triangle. The transformations by which one of the small shaded triangles at the centre of fig. 23 is carried into the other three shaded triangles at the centre correspond on the sphere to rotations through angles $\pi/2$, π, $3\pi/2$ about an axis connecting two

* Schwarz, *loc. cit.*; Klein, *Math. Ann.*, **9** (1875), p. 183.

† I am indebted to Professor Crum Brown for the construction of the model of which this figure is a photograph.—L. R. F.

opposite vertices of the octahedron. The transformations by which one of these triangles is carried into the other two shaded triangles lying in the same large triangle correspond to rotations through angles $2\pi/3$, $4\pi/3$ about an axis through the centres of opposite faces of the octahedron. In each of these rotations the octahedron is transformed into itself. Similarly we can show that the linear transformation by which a shaded

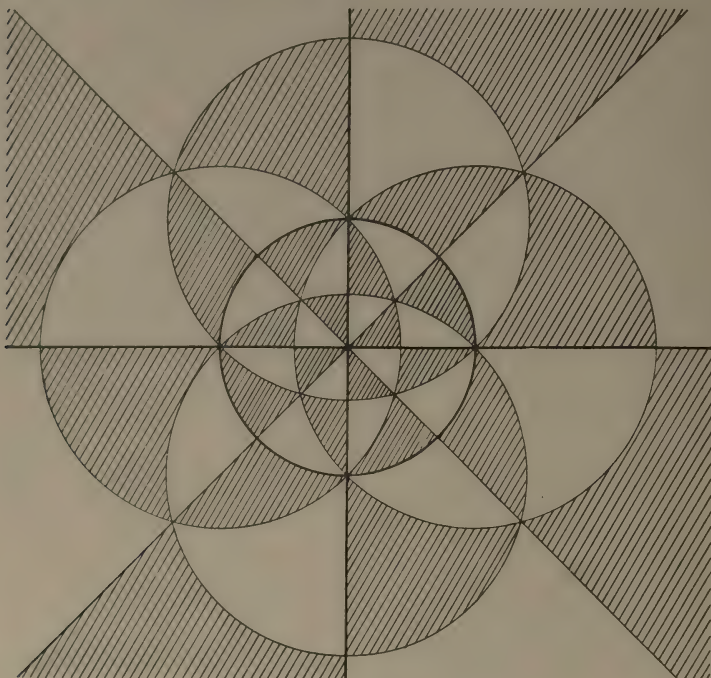

FIG. 23.

triangle is transformed into any of the other twenty-three shaded triangles corresponds to a rotation of the sphere by which the octahedron is transformed into itself. Hence, including the case of no rotation (the identical transformation), *there is a group of twenty-four rotations by which the octahedron is transformed into itself.*

But we can say more. The inversions by which a shaded triangle is carried into an adjacent unshaded triangle, and in fact into any shaded triangle, carry each of the eight large

triangles either into itself or into some other large triangle. On the sphere these inversions are reflections in diametral planes by which the octahedron is transformed into itself. The planes in question are the planes of symmetry of the solid, each of which passes through either four vertices or the mid-points of four faces. Since a shaded triangle may be carried into any one of the unshaded regions, there are twenty-four of these reflections. They do not form a group, since the succession of

FIG. 24.

two reflections is not a reflection but a rotation. But *the twenty-four reflections together with the twenty-four rotations mentioned above constitute a group of transformations by which the octahedron is transformed into itself.*

Suppose that, instead of grouping the forty-eight triangles into eight large equiangular triangles each containing six, we now group them into six quadrilaterals each containing eight triangles. Take the eight triangles at the centre of fig. 23 as one quadrilateral. The four quadrilaterals arising by inversions in its sides together with that composed of the eight triangles meeting at infinity constitute the other five. By projection on the sphere the central figure becomes a spherical quadrilateral

of four equal sides. The same is true of the others, since they are reflections of the first quadrilateral in diametral planes. Replacing the arcs by their chords, we get the edges of an inscribed cube. The radii to the vertices of the cube pass through the mid-points of the faces of the octahedron; and the radii to the vertices of the octahedron pass through the mid-points of the faces of the cube. The two solids have the same planes of symmetry. Because of this fact (or we can show it directly as before) *the twenty-four reflections and twenty-four rotations of the preceding group transform the cube into itself.*

The remaining cases of III. furnish the groups of rotations and reflections for the other regular solids. In (i.) $\frac{1}{2}$, $\frac{1}{2}$, $1/n$, we have the dihedron, or n-gon, a regular "solid" of two faces. Each of the faces is a regular polygon of n sides. The two faces coincide, and the figure has zero volume. In (ii.) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{3}$, we have the tetrahedron; and in (iv.) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, we have the ikosahedron and dodekahedron. In each case the solid is transformed into itself by a group of N/2 rotations and N/2 reflections in planes of symmetry, where N is the total number of triangles.*

* In Klein's *Lectures on the Ikosahedron* will be found a treatment of all the solids. Special attention is paid to case (iv.), the group of sixty rotations and sixty reflections being used in investigating the general equation of the fifth degree.

CHAPTER V

NON-EUCLIDEAN GEOMETRY

33. Absolute, or Cayleyan, Geometry.—The system of geometry to be described in this section was introduced by Cayley.* It is a geometry in which the terms "distance" and "angle" are shorn of their usual meanings and are given new definitions. These definitions are made to depend upon a chosen fixed conic called the *absolute.*

Let P, Q be two points; and let the line PQ meet the

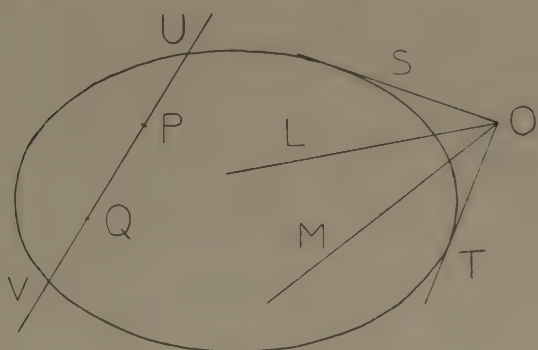

FIG. 25.

absolute in the points U, V (fig. 25). Then the *distance* PQ is defined to be proportional to the logarithm of the cross ratio of the four points P, Q, U, V; that is,

$$\text{Dist. PQ} = k \log \frac{\text{PU . QV}}{\text{PV . QU}}.$$

Let L, M be two lines intersecting at O; and let S, T be the tangents from O to the absolute. The *angle between the lines*

* *Sixth Memoir on Quantics*, Phil. Trans., **149**, 1859, reprinted in Cayley's *Scientific Papers*, vol. ii.

L, M is defined to be proportional to the logarithm of the cross ratio of the four lines L, M, S, T; that is,

$$\text{Angle LOM} = k \log \frac{\sin \text{LOS} \sin \text{MOT}}{\sin \text{LOT} \sin \text{MOS}}.$$

We verify directly from the definitions that if R is a point on the line PQ, Dist. PQ + Dist. QR = Dist. PR; and that if N is a line through O, angle LOM + angle MON = angle LON. We can show, in fact, that all the axioms of ordinary Euclidean geometry hold with the exception of the parallel postulate. We note also from the definition that if P approaches U the distance PQ becomes infinite, and that if L approaches S the angle LOM becomes infinite. If O approaches the absolute, S and T approach coincidence and the angle LOM approaches 0; that is, the angle between two lines meeting at infinity is zero. L and M are then said to be parallel.

34. The Three Types of Geometry.—We choose for the absolute a conic whose equation has real coefficients. There are then three types of geometry, as follows:—

(1) The absolute has a real locus, and divides the plane into two parts. This is the case of the ordinary real hyperbola, parabola, or ellipse. The geometry is called *hyperbolic*.

(2) The absolute has no real locus. The geometry is called *elliptic*.

(3) The absolute is a degenerate conic consisting of a real double line. The plane is not divided into two parts. The geometry is called *parabolic*. This type is the limiting case separating the two preceding.

We easily discover many of the fundamental facts of the various types. If the geometry is hyperbolic, two lines can be drawn through a given point parallel to a given line. Thus in the given figure where we consider points interior to the ellipse the two lines through U,V and the given point are parallel to PQ. In parabolic geometry there is one parallel to a given line through a given point, since the two intersections with the absolute coincide. In elliptic geometry there are none, since a given real line has no real intersection with the absolute. It can also be shown without difficulty that in hyperbolic geometry the sum of the angles of a triangle is less than two right angles *

* The sum of the angles about a point being called four right angles.

(the sum is zero if the vertices of the triangle lie on the absolute); in parabolic geometry the sum is equal to two right angles; in elliptic geometry the sum is greater than two right angles. The plane of elliptic geometry is characterised by the fact that there are no infinite points, for the infinite points lie on the absolute which is imaginary. It follows from this that each straight line is closed and of finite length.

The homogeneous equation of the absolute can by means of a projection be taken in the form

$$e(x^2+y^2)-z^2=0,$$

and the geometry is hyperbolic, elliptic, or parabolic according as e is positive, negative, or zero. The Cayleyan distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ we find, on forming the cross ratio of the definition, to be

$$k\log\frac{e(x_1x_2+y_1y_2)-z_1z_2+\sqrt{\{[e(x_1x_2+y_1y_2)-z_1z_2]^2-[e(x_1^2+y_1^2)-z_1^2][e(x_2^2+y_2^2)-z_2^2]\}}}{e(x_1x_2+y_1y_2)-z_1z_2-\sqrt{\{[e(x_1x_2+y_1y_2)-z_1z_2]^2-[e(x_1^2+y_1^2)-z_1^2][e(x_2^2+y_2^2)-z_2^2]\}}}.$$

We shall now show that when $e=0$ we have the usual Euclidean definition of distance. When e approaches zero, the formula just found approaches zero unless k, which need be constant only for a fixed value of e, is made a function of e. Taking $k=-1/2\sqrt{e}$ we get for the distance when e approaches zero

$$\text{Lim.}\ \frac{-1}{2\sqrt{e}}\log\left\{1+\frac{2\sqrt{\{[e(x_1x_2+y_1y_2)-z_1z_2]^2-[e(x_1^2+y_1^2)-z_1^2][e(x_2^2+y_2^2)-z_2^2]\}}}{e(x_1x_2+y_1y_2)-z_1z_2}\right\}$$

$$=\text{Lim.}\ \frac{-1}{\sqrt{e}}\frac{\sqrt{\{[e(x_1x_2+y_1y_2)-z_1z_2]^2-[e(x_1^2+y_1^2)-z_1^2][e(x_2^2+y_2^2)-z_2^2]\}}}{e(x_1x_2+y_1y_2)-z_1z_2}$$

$$=\frac{\sqrt{\{(x_1z_2-x_2z_1)^2+(y_1z_2-y_2z_1)^2\}}}{z_1z_2},$$

which is the ordinary Euclidean formula for distance. A similar treatment of angle can be made from the tangential equation of the absolute, the result being that the case $e=0$ yields the ordinary definition of angle. When $e=0$ the absolute is $z^2=0$. *Euclidean geometry is thus a special case of absolute geometry, the absolute being the line at infinity counted twice.*

35. Representation of Non-Euclidean Straight Lines by Circles in the Plane. The Elliptic Case.—It is our aim to arrive at an interpretation of certain facts concerning Fuchsian and other groups by means of the ideas of non-Euclidean geometry. In pursuance of this plan it will now be shown how

the straight lines of non-Euclidean geometry can be represented by circles in the plane. We shall work out the formulæ for distance only, merely stating the results in the case of angle. The treatment of angle is very similar to that of distance, the tangential equation of the absolute being used.

The familiar example of a geometry in which the sum of the angles of a triangle is greater than two right angles is geometry on the surface of a sphere. Angle here has its ordinary meaning; and the distance between the points is the length of the arc of the great circle joining them. We shall see how this geometry arises from elliptic absolute geometry.

Let $x^2+y^2+1=0$ be the equation of the absolute. Let the plane of the absolute be placed tangent to the sphere $X^2+Y^2+Z^2=1$, at the point (0, 0, 1), the x-and y-axes of the plane being parallel to the X- and Y-axes of space. Now let the plane be projected on the sphere, the centre of the sphere being the centre of projection. [The projection from its centre of a sphere on a plane is called a *gnomonic projection*. The projection we have just made is the inverse of this transformation.] The straight lines of the plane project into great circles on the sphere, since the plane through the straight line and the centre of projection cuts the sphere in a great circle. If now we transform the formulæ for distance and angle, replacing the coordinates of points in the plane by their values in terms of the coordinates of the corresponding points on the sphere, we shall have a consistent system of elliptic geometry on the sphere in which straight lines are represented by great circles.*

Let us see what these formulæ become. If $P(x, y)$ is a point in the plane and $P'(X, Y, Z)$ the corresponding point on the sphere, we have, since P, P′, and the origin lie in a line, $x:y:1=X:Y:Z$. X, Y, Z can then be used as the homogeneous coordinates of the point P. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane, the corresponding points on the sphere being $P'(X_1, Y_1, Z_1)$ and $Q'(X_2, Y_2, Z_2)$. The absolute distance PQ is, using the formula of the preceding section—

$$k\log\frac{X_1X_2+Y_1Y_2+Z_1Z_2+\sqrt{\{(X_1X_2+Y_1Y_2+Z_1Z_2)^2-(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)\}}}{X_1X_2+Y_1Y_2+Z_1Z_2-\sqrt{\{(X_1X_2+Y_1Y_2+Z_1Z_2)^2-(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)\}}}.$$

* The connection of elliptic geometry with ordinary spherical geometry is due to Klein, *Math. Ann.*, **4** (1872) and **6** (1873).

Since (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) are the non-homogeneous coordinates of a point on the sphere, we have $X_1^2+Y_1^2+Z_1^2=X_2^2+Y_2^2+Z_2^2=1$. Let θ be the length of the arc of a great circle joining P′ and Q′. Then $\cos\theta=X_1X_2+Y_1Y_2+Z_1Z_2$. The formula then becomes

$$\text{Dist. PQ} = k \log \frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} = k \log e^{2i\theta} \doteq 2ik\theta.$$

Putting $k=-\frac{1}{2}i$, *we have the Cayleyan distance equal to the spherical arc joining* P′Q′. A similar treatment of angle will show that *the Cayleyan angle between two lines is equal to the angle (in the ordinary sense) between the corresponding great circles on the sphere.*

If now we project the sphere stereographically on a plane, the great circles become circles in the plane. If we let angle have its ordinary meaning and define the length of a curve to be that function of the curve which is equal to the length of its stereographic projection on the sphere—we do not need to derive the function,—we have a system of elliptic geometry in which straight lines are represented by the circles just found.

It is easy to see what these circles are. If the projection is made from the point (0, 0, 1) on the plane $Z=0$, the great circle common to the sphere and plane, $X^2+Y^2=1$, is unchanged. Each great circle on the sphere intersects this circle at opposite ends of a diameter, and the same is true of the projection of the great circle. The straight lines are then represented by the two-parameter family of circles $X^2+Y^2+AX+BY=1$.

36. The Hyperbolic Case.—If the absolute is real, we can by a projection take it to be the circle $x^2+y^2=1$. We shall now project the plane on the sphere $X^2+Y^2+Z^2=1$ as follows. Place the plane of the absolute in coincidence with the plane $Y=0$, the origin in the plane being at the point (0, 0, 0) and the x- and y-axes being parallel respectively to the X- and Z-axes. We take as centre of projection the point in homogeneous coordinates (0, 1, 0, 0)—the point where the Y-axis meets the plane at infinity. The projections of a point in the plane are the two points in which a line through the given point perpendicular to the plane $Y=0$ meets the sphere. The straight lines in the plane project into circles on the sphere which are orthogonal to the plane $Y=0$.

Now let this system of circles be projected stereographically from the point (0, 0, 1) on the plane $Z=0$. The circle in which the plane $Y=0$ cuts the sphere (*i.e.* the absolute, which was unchanged by the previous projection) is projected into the line $Y=0$; and the system of circles is projected into the circles orthogonal to this line. Hence, if the definitions of distance and angle be carried over, *we have a representation of hyperbolic geometry in the plane in which straight lines are represented by circles orthogonal to a fixed line.*

We shall now determine the formula for distance. The equations connecting the points on the Cayleyan plane with those on the sphere are evidently $x=X$, $y=Z$. Let the coordinates in the plane on which the sphere is stereographically projected be ξ, η. By considering the projections on the coordinate axes of the line through a point (ξ, η) of the plane, its corresponding point (X, Y, Z) on the sphere and the point (0, 0, 1), we derive the relations $\xi:\eta:1=X:Y:1-Z$, where $X^2+Y^2+Z^2=1$. From these we get the equations of the transformation—

$$\xi=x/(1-y), \quad \eta=\sqrt{(1-x^2-y^2)}/(1-y).$$

It will be most convenient to get the formula for distance in the form of an integral. Let $P(x, y)$, $Q(x+\Delta x, y+\Delta y)$ be neighbouring points in the xy-plane, $P'(\xi, \eta)$, $Q'(\xi+\Delta\xi, \eta+\Delta\eta)$ being the corresponding points in the $\xi\eta$-plane. Substituting $(x, y, 1)$ and $(x+\Delta x, y+\Delta y, 1)$ for (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively in the formula of Section 34, we have, e now being 1—

$$\text{Dist. PQ}=k\log\frac{x^2+y^2-1+x\Delta x+y\Delta y+\sqrt{\{[x\Delta x+y\Delta y]^2-[x^2+y^2-1][(\Delta x)^2+(\Delta y)^2]\}}}{x^2+y^2-1+x\Delta x+y\Delta y-\sqrt{\{[x\Delta x+y\Delta y]^2-[x^2+y^2-1][(\Delta x)^2+(\Delta y)^2]\}}}.$$

Dropping infinitesimals of higher order than the first, this can be written—

$$\begin{aligned}\text{Dist. PQ}&=k\log\left\{1+\frac{2\sqrt{\{[x\Delta x+y\Delta y]^2-[x^2+y^2-1][(\Delta x)^2+(\Delta y)^2]\}}}{x^2+y^2-1}\right\}\\&=2k\sqrt{\left\{\left(\frac{x\Delta x+y\Delta y}{x^2+y^2-1}\right)^2-\frac{(\Delta x)^2+(\Delta y)^2}{x^2+y^2-1}\right\}}\\&=2k\sqrt{\left\{\left[\frac{x\Delta x+y\Delta y}{x^2+y^2-1}+\frac{\Delta y}{1-y}\right]^2+\left[\frac{\Delta x+x\Delta y/(1-y)}{\sqrt{(1-x^2-y^2)}}\right]^2\right\}}.\end{aligned}$$

From the form in which we have written the last equation we verify at once from the equations of the transformation that

$$\text{Dist. PQ} = 2k\sqrt{\{(\Delta\xi/\eta)^2 + (\Delta\eta/\eta)^2\}} = 2k\sqrt{\{(\Delta\xi)^2 + (\Delta\eta)^2\}}/\eta.$$

If now we use the customary complex variable $z = \xi + i\eta$, this becomes, putting $2k = 1$, $|\Delta z|/\eta$; and this is the value to be used for the infinitesimal distance P′Q′. *The length of a curve in the* $\xi\eta$*-plane is the value of the integral* $\int |dz|/\eta$ *taken along the curve*;* and the shortest distance between two points A and B is the integral $\int_A^B |dz|/\eta$, the path of integration being the circle through the points orthogonal to the ξ-axis.

A consideration of angle shows that, as in the elliptic case, *angle has its ordinary meaning.*

37. Application to Groups of Linear Transformations.—Let us return to the Cayleyan plane. We shall call a transformation of the plane by which both distances and angles are unchanged a *displacement.* Since distances are to be unchanged, an infinitely distant point must remain at infinity; that is, the absolute is transformed into itself. It follows also that lines are transformed into lines; for if P, Q, R lie on a line, Dist. PQ + Dist. QR = Dist. PR, and we shall have for the transformed points Dist. P′Q′ + Dist. Q′R′ = Dist. P′R′, which is true only if P′, Q′, R′ are collinear. The displacement is then a collineation—

$$x' = \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \quad y' = \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}.$$

The invariance of the absolute imposes five conditions on the coefficients. A collineation leaves invariant the cross ratio of four points on a line or of four lines through a point; and we observe from the definitions of distance and of angle that the collineations which transform the absolute into itself preserve distance and angle save possibly for sign.

Now let us consider the representation of the hyperbolic plane in which the straight lines are represented by circles orthogonal to the real axis. We found that the absolute is transformed into that axis. The displacements are the directly

* Poincaré, *Acta Mathematica*, vol. i. pp. 7, 8.

conformal transformations which leave the real axis invariant and transform circles orthogonal to the real axis into circles orthogonal to that axis; that is, the linear transformations. The inversely conformal transformations leaving the absolute invariant are the inversions in circles orthogonal to the real axis. They preserve lengths but change the signs of angles. The transformation amounts, to use the Euclidean analogy, to rotating the plane through 180° about one of its lines.

In this hyperbolic geometry, the fundamental region of a Fuchsian group is a rectilinear polygon. The transformations of the group are all displacements. The various transforms of the fundamental region which partition the plane are equal in all respects, and any one can be brought into coincidence with any other without alteration of shape or magnitude Certain facts of previous proofs become obvious; for example, that only a finite number of regions will be contained within an area not extending to the real axis; and that the regions will cluster in infinite number along the axis.

In the elliptic case we projected the plane on the sphere in such a manner that great circles play the rôle of straight lines. The displacements are the rigid motions of the sphere; the transformations which preserve lengths but change the signs of angles are the reflections in diametral planes. [It is easy to show that in the transformation we made, the absolute is projected into the circle at infinity, which is unchanged by the rotations and reflections mentioned.]

The three kinds of triangles that arose in connection with the triangle functions can be interpreted in the light of the three kinds of geometry. In each case, all the triangles of a network are equal to each other in magnitude when the appropriate type of geometry is used. When $\lambda+\mu+\nu=1$ the sum of the angles of the triangle is two right angles and the geometry is Euclidean (fig. 19). When $\lambda+\mu+\nu<1$ the sum of the angles is less than two right angles, and the geometry is hyperbolic (fig. 20). When $\lambda+\mu+\nu>1$ the sum of the angles is greater than two right angles and the geometry is elliptic (fig. 23). In each case a shaded triangle can be brought into coincidence with any other shaded triangle by a displacement or into coincidence with any unshaded triangle by a displacement together with a turning over.

The Fuchsian groups furnish us with discontinuous groups of collineations which leave a real conic invariant. For, each transformation of the group can be expressed as a transformation of the points in the Cayleyan plane by means of the equations defining the transformation from one plane to the other. The circular-arc fundamental region in the complex plane is transformed into a rectilinear polygon in the Cayleyan plane, which is a fundamental region for the group of collineations. The division of the plane for such a group of collineations is shown in fig. 26. The angles of each triangle are $\pi/3$, $\pi/3$, 0. The figure is, in fact, the division for the modular group carried over to the Cayleyan plane by means of the equations of Section 36.

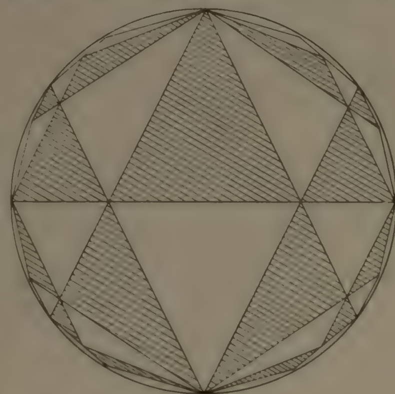

Fig. 26.

It is possible also to discover independently discontinuous groups of collineations which leave a certain conic invariant, and from these to derive new discontinuous groups of linear transformations. Studies of the subject from this point of view have been made by Poincaré and by Fricke.*

* Poincaré, *Journal de Math.*, **3** (1887), 405; Fricke, *Math. Ann.*, **38** (1890), 50, 461.

CHAPTER VI

UNIFORMISATION

38. Riemann's Surfaces.—The functions of greatest simplicity in analysis are the single-valued functions; to such the simple theorems of the theory of functions can be applied without reservation. In the case of functions which are many-valued, there are complications owing to the fact that it is necessary to deal with a particular one of the many branches of the function. We shall recall here a device introduced by Riemann in which the plane of the independent variable is replaced by a multiplicity of planes in such a manner that on the resulting configuration the function is single-valued.

Consider first the function $w^2=x$. Each value of x, excepting the points 0 and ∞, yields two values of w. Let $x_0=\rho_0 e^{i\theta_0}$ be a point in the x-plane, ρ_0 being the distance from the origin 0 to x_0 and θ_0 the angle between $0x_0$ and the real axis. The two values of w are $w_1=\sqrt{\rho_0}e^{i\theta_0/2}$ and $w_2=-w_1=\sqrt{\rho_0}e^{i\theta_0/2+\pi i}$. Let us fix our attention on w_1 while x is varied. If x describes any closed path not encircling the origin, both the distance from the origin and the angle return to their original values; that is, w returns to the value w_1. If, however, x passes once around the origin, the angle is increased or diminished by 2π, and w_1 acquires the value w_2; another circuit of the origin brings the value back to w_1. Now let us superpose on the x-plane a second x-plane infinitesimally near it, each point of which represents the same value of x as the point immediately below it. Let the two planes be cut along curves extending from the origin to infinity, say along the negative half of the real axis. Join the part above the axis in each sheet to the part below the axis in the other sheet. In the resulting surface a moving point x passes from one sheet into the other on crossing the negative half of the real axis. Now let us consider the variation of w, starting with the value w_1 at x_0 in the original plane. If x makes a circuit not surrounding the origin, either x remains always in the original plane, or if it crosses the negative part of the real axis into the second plane it recrosses into the original plane, and returns to x_0 with the original value w_1. If, however, x makes a circuit of the origin, it returns to x_0 in the second plane with the value w_2. In general whatever path x makes returning to x_0 the function has the value w_1 if x is in the original plane, and w_2 if x is in the second plane. *On this configuration, the "Riemann's surface," the function is single-valued.* The points 0 and ∞ at which the line of juncture of the sheets terminates are called *branch points.*

This simple example exhibits the essential features of the Riemann's surface for the general algebraic function. Let w be defined as a function of x by the irreducible * equation $\Phi(w, x) = P_0(x)w^m + P_1(x)w^{m-1} + \ . \ . \ . \ + P_m(x) = 0$, where $P_i(x)$ is a polynomial in x. For a given value of x the coefficients P_i are fixed and the solution of the equation gives m values of w. The fact is, as the preceding example would lead us to expect, that w is a single-valued function of x on a Riemann's surface composed of m sheets joined along certain lines terminating in branch points. Similarly the same equation defines x as a function of w single-valued on a Riemann's surface covering the w-plane, the number of sheets equalling the degree of the equation in x.†

39. Genus.—The study of the algebraic equation into which a given algebraic equation $\Phi(w, x) = 0$ can be transformed has led to the important conception of genus. If we make a projection

$$w' = \frac{a_1 w + b_1 x + c_1}{a_3 w + b_3 x + c_3} \qquad x' = \frac{a_2 w + b_2 x + c_2}{a_3 w + b_3 x + c_3}$$

there are certain numbers connected with the curve ‡ which are unchanged. For example, the degree of the transformed curve, its class, and the number of its double points are the same as in the original curve.

Consider now the most general kind of transformation by which each point of the given curve is transformed into one and only one point of the transformed curve, and no two points of the given curve are transformed into the same point. In other words, the points on the Riemann's surfaces of the two curves are to correspond in a one-to-one manner. It is known that the most general such transformation is one in which the transformation from the points on one curve to the points on the other are expressible rationally: $w' = R_1(w, x)$, $x' = R_2(w, x)$ and $w = R_1'(w', x')$, $x = R_2'(w', x')$. Owing to its form this transformation is called *birational.*

* *I.e.*, not breaking up into factors of lower degree.

† There exists a Riemann's surface for any function $w = f(x)$; but the non-algebraic cases are characterised by the fact that the number of sheets, either for $w = f(x)$ or for the inverse function $x = f^{-1}(w)$, is infinite.

‡ It is helpful to interpret our statements by means of plane curves: although in general the coefficients in the equation $\Phi(w, x) = 0$ are complex and the curve has no real locus.

It is not necessary that it be possible to solve the equations $w'=R_1(w, x)$, $x'=R_2(w, x)$ for w, x rationally in term of w', x'. But it must be possible to do so by using the fact that w, x are connected by the algebraic equation $\Phi(w, x)=0$. The resulting rational equations are not the general inverse transformation, but an inverse transformation which holds only for the points on the two curves. An example will make this clearer. Consider the curve $w^2=x^3-x$, and the transformation $w'=w$, $x'=x^2$ which transforms it into $w'^4=x'(x'-1)^2$. The general inverse transformation $w=w'$, $x=\sqrt{x'}$ is not rational; but if we make use of the equation of the curve the inverse transformation can be written rationally thus: $w=w'$, $x=w'^2/(x'-1)$. The transformation is thus birational. An exceptional case occurs for this transformation if the given curve is symmetrical with respect to the w-axis, as $w=x^2$. Then two points on the given curve yield the same point on the transformed curve. The transformation is not birational—we are unable by using the equation $w=x^2$ to get an inverse function involving only rational functions.

If the inverse is rational without using the equation of the curve the transformation is called a *Cremona* transformation; for example, $w'=x/w$, $x'=1/x$, which has the inverse $w=1/w'x'$, $x=1/x'$. The Cremona transformation is thus a special case of the birational transformation.

We readily find that in the case of the birational transformations neither the degree, nor the class, nor the number of double points is unchanged. There is, however, an important invariant number. Let n, n' be the degrees of the original and the transformed curves, and d, d' the numbers of their double points. In all cases $(n'-1)(n'-2)/2-d'=(n-1)(n-2)/2-d$. This invariant number, $(n-1)(n-2)/2-d$, is called the *genus** of the algebraic relation. It is usually represented by the letter p. The maximum number of double points which an irreducible curve of the nth degree can have is $(n-1)(n-2)/2$; hence the genus of a curve is the number of double points by which it fails to attain this maximum.†

40. Connectivity of Surfaces.—We can take an entirely different view of the genus of an algebraic equation, and one more useful for our purposes, by considering the properties of the Riemann's surface quite apart from the equation connected with it. It is necessary first to state some general results concerning what is termed the connectivity of a surface.

A surface is said to be *connected* if we can join any point of the surface to any other of its points by a curve lying entirely within the surface. For example, the surface of a sphere, a

* In curve theory it is commonly called *deficiency*. The French term is *genre*, the German, *Geschlecht*. *Genus* was introduced in Riemann's *Theorie der Abel'schen Funktionen*, Crelle's J., **54** (1851). The term *deficiency* is due to Cayley.

† Multiple points of higher order than simple double points are to be counted as the equivalent number of double points.

plane surface bounded by a circle, a plane circular ring bounded by two concentric circles are connected. We shall assume that the surface has a boundary; if it is a closed surface, such as the sphere, we shall make a small hole in the surface to provide an initial boundary.*

Now let us cut the surface along a curve beginning and ending in the boundary. If, however this cut be made, the surface is divided into two pieces, the surface is said to be *simply connected*; otherwise it is *multiply connected.* The first and second of the surfaces just mentioned are simply connected; the third is multiply connected, since a cut extending from one boundary circle to the other does not cut the surface into two pieces. If one cut renders the surface simply connected, the surface is said to be *doubly connected*; otherwise we can make a second cut beginning and ending in the boundary (which now includes the two sides of the first cut). In general, if n cuts are necessary to make the surface simply connected, the surface is $(n+1)$-ply connected, or is *of connectivity* $n+1$.

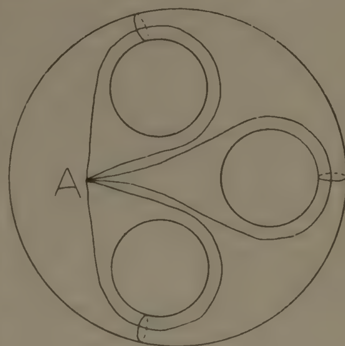

FIG. 27.

It is clear that if a surface be continuously deformed in any way, or be transformed in a one-to-one and continuous manner into any other surface, its connectivity is unchanged; for the cuts which render it simply connected will be transformed into cuts rendering the new surface simply connected.

It can be proved without difficulty that a surface with a single boundary, such as a closed surface with a small hole in it, is of odd connectivity. We can then represent its connectivity by $2p+1$, where p is an integer. The number p is called the *class* of the surface. It can be shown that a surface can be transformed in a continuous and one-to-one manner into any other surface of the same class.†

* We shall not consider in this section the so-called one-sided surfaces. The often-quoted example of such a surface is that formed by a strip of paper whose ends are joined after one of them has been given a twist through 180°.

† In fact, a surface can be transformed into any other of the same connectivity if they have the same number of boundaries.

The simplest example of a surface of class p is that of a closed surface with p holes through it. In fig. 27 is shown such a surface of class 3. The initial boundary is at A; and the six cuts which render the surface simply connected are shown.

Now let us consider the Riemann's surface in the light of the preceding remarks. To avoid considering the points at infinity, let it be projected stereographically on the sphere. The surface is a closed surface consisting of several spherical sheets; and it can be shown that *the class of this surface is equal to the genus of the algebraic relation connected with it.** In view of this fact it becomes evident intuitively why the genus is unchanged by the birational, or one-to-one, transformations.

41. Genus of the Fundamental Region of a Group.—In Section 21, Theorem F, we found that two functions, say w and x, automorphic with respect to a group are connected by an algebraic relation $\Phi(w, x) = 0$. Each point z of the fundamental region yields a pair of values $w(z)$, $x(z)$ satisfying the equation; and in general no two points of the region yield the same pair of values. In other words, each point of the fundamental region corresponds to one, and only one, point of the Riemann's surface; and conversely each point of the surface is derived from one, and only one, point of the fundamental region. Since w and x are automorphic, congruent values of z correspond to the same point on the surface. Thus two congruent edges of the fundamental region correspond to the same curve on the surface.

Let us now form the fundamental region into a closed surface by bringing congruent edges together, each point being brought into coincidence with its congruent point. We thus get a closed surface which corresponds point for point with the Riemann's surface. *The genus (class) of this surface is the same as that of the Riemann's surface. We define the genus of the fundamental region to be the genus of this closed surface formed by bringing its congruent edges together.*

In fig. 28 we give an example of this process. In (I) is shown a fundamental region, the congruent sides being connected by arrows. Bringing the sides B and H,

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* For the proof of this fact in an important special case see Forsyth, *Theory of Functions*, p. 355. An excellent treatment of the connectivity of surfaces will be found in Chapter XIV. of the same work.

and D and F into coincidence, we get (II). A and E are now closed curves, and it is necessary to bring the opposite ends of G and C together, as in (III), so that the remaining sides can be joined without tearing. Making the juncture, we get the surface in (IV), which is of genus 2. We note that the eight vertices form a cycle; and these are brought into coincidence on the surface.

Since the points of this closed surface formed from the fundamental region correspond in a one-to-one manner with the points of the Riemann's surface for $\Phi(w, x)=0$, and since neighbouring points on one surface correspond to neighbouring points on the other, it follows that this new surface can be used as one on which to represent the pairs of values (w, x) satisfying the equation. In other words, it would serve as a Riemann's surface for the algebraic equation quite as well as the original

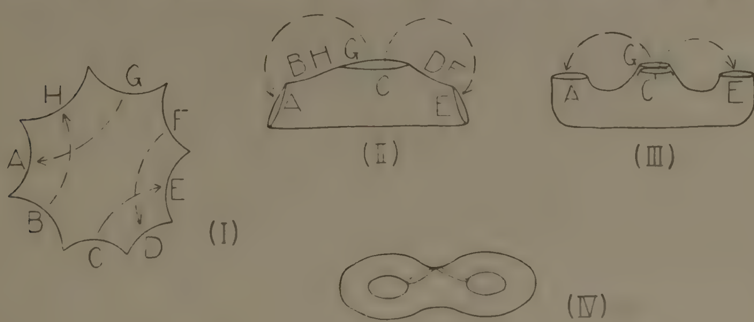

FIG. 28.

surface composed of several plane sheets. Viewed in this light, we see that *the fundamental region before being joined into a closed surface is the Riemann's surface dissected into a simply connected region, or at least partially dissected.* [The fundamental region is not always simply connected. See fig. 12, where some further cuts are necessary to make the region simply connected.] This use of the fundamental region as a dissected Riemann's surface is particularly useful because of the fact that both w and x are single-valued analytic functions of the variable z of the fundamental region.

42. The Problem of Uniformisation.—The theory of single-valued functions is both simpler and more developed than that of many-valued functions; and if we can express each of the variables of the many-valued functional relation as single-valued functions of a third variable a substantial contribution

to the study of the function is made. This is the problem of uniformisation.

Let $w=f(x)$ be a many-valued function. We wish to express x and w as single-valued functions of a third variable, $x=\phi(z)$, $w=\psi(z)$, such that for each value of z for which both functions are defined the corresponding values of x and w satisfy the equation $w=f(x)$, and each pair (w, x), satisfying the equation is given by one point z at least. Looked at from the standpoint of curve theory, it is the problem of expressing the curve in parametric form by means of single-valued functions. From another point of view it is the problem of finding two single-valued functions which map a portion of the z-plane in a one-to-one and generally conformal manner upon the Riemann's surface of the functions $w=f(x)$.

We consider here particularly the algebraic curves. It is well known that those of genus 0 can be uniformised by means of rational functions. For example, the two-valued function $w^2=1-x^2$ is uniformised by the functions $x=2z/(1+z^2)$, $w=(1-z^2)/(1+z^2)$. These functions map the whole z-plane upon the two-sheeted Riemann's surface.

The curves of genus 1 can be uniformised by means of doubly periodic functions. We can employ functions of the form $x=\phi[\wp(z), \wp'(z)]$, $w=\psi[\wp(z), \wp'(z)]$, where $\wp(z)$ is the Weierstrassian function and where ϕ and ψ are rational functions of their arguments. The functions map each of the period parallelograms on the Riemann's surface. [See fig. 7. We observe that by joining congruent edges of the period parallelogram the resulting surface is of class 1.]

43. Uniformisation by Means of Automorphic Functions.—The facts brought out in preceding sections lead us to hope that the general algebraic curve can be uniformised by means of simple automorphic functions. For we found that between two such functions, $w(z)$, $x(z)$, there exists an algebraic relation $\Phi(w, x)=0$; and this is obviously uniformised by the functions $x=x(z)$, $w=w(z)$. But there was no indication as to whether all algebraic relations could arise in this way. It is the fact, however, that there are no exceptions. It has been established, by means of a general existence theorem, that—

THEOREM J.—*Any algebraic curve can be uniformised by*

means of Fuchsian functions, the fundamental region of the group to which the functions belong lying within the principal circle.

Also a much more general theorem has been established. Let $w=f(x)$ be a multiple-valued function of any kind. It has been proved that it can be uniformised by means of automorphic functions; that

THEOREM K.—*The most general many-valued function can be uniformised by means of automorphic functions of a group with a principal circle. The functions have no other singularities than poles within the principal circle, and the circle is a natural boundary for one at least of the functions.*

Let us examine the statements in this theorem. If the function to be uniformised is transcendental, it is clear that the uniformising functions cannot be the simple automorphic functions studied in Chapter III., for these functions are connected by an algebraic relation. Since the uniformising functions have only polar singularities within the region, it follows that the fundamental region of the group contains portions of the principal circle on its interior, or at least has vertices on the principal circle, where one at least of the functions has singularities other than poles. It is necessary, in fact, in establishing the theorem to admit the single transformation $z'=z$ as a possible group for this case, and the fundamental region for this group is the whole plane.

The proofs of these theorems, while simple enough in conception, involve considerable detail in the execution; and they cannot be given here. So far as I am aware, the only complete and simple presentation is to be found in Osgood's *Funktionentheorie.**

The automorphic functions of the types just mentioned are not the only ones which can be used in the uniformisation of functions. We shall presently give an example of the uniformisation of algebraic functions by means of functions whose group possesses a fundamental region not confined to the interior of

* Vol. i., 2nd ed., chapter xiv. The treatment is based on a series of papers by Koebe in the *Göttinger Nachrichten*, 1907-9, and in the *Mathematische Annalen*, vols. **67** (1909) and **69** (1910). Further papers by Koebe have since appeared in the latter publication. It was Klein who first saw the possibility of arriving at a general existence theorem. Cf. *Math. Ann.*, **21** (1883), p. 162, and Abschnitt iii., § 10.

the principal circle. The uniformisation can also be accomplished by functions whose group preserves no circle invariant. But functions of the kind mentioned in the theorem would seem to be usually the most simple. There is one advantage in the fact that a fundamental region within the principal circle is always simply connected.

The advantages of uniformisation are many. In the study of functions on a Riemann's surface—functions of the form $f(w, x)$, where w and x satisfy an algebraic equation $\Phi(w, x)=0$—the investigations are lacking in simplicity owing to the necessity of considering the many exceptional points. The points at which w or x become infinite involve one kind of series expansion, the branch points another, both differing from that at ordinary points. On the fundamental region in the plane of the uniformising variable these exceptional points do not occur, for the fundamental region has no infinite points and no branch points.

Not least among the simplifications introduced by uniformisation is to be found in the treatment of Abelian integrals—integrals of the form $\int R(w, x)dx$, where $\Phi(w, x)=0$, and R is a rational function. To take a simple example from the calculus: the integral $\int dx/\sqrt{(1-x^2)}$—that is, $\int dx/w$, where $w^2=1-x^2$—is most easily found by putting $x=\sin z, w=\cos z$; in other words, by uniformising the curve $w^2=1-x^2$ by means of automorphic functions.

In case the genus of the surface is greater than 0 the integrals are always infinitely many-valued on the surface; when certain closed paths are described, the integral is changed by the addition of a constant. The integrals become single-valued (provided they have no logarithmic singularities) when the surface is dissected into a simply connected piece. But in the plane of the uniformising variable where the fundamental region is simply connected these integrals are single-valued. A path on the Riemann's surface by which the integral returns to its original position with an added modulus corresponds to a path in the plane of the uniformising variable which terminates, not in the point of departure, but in a point congruent to it. The integral is then a single-valued function which changes by an additive constant (which may be zero) in passing from any point

to a congruent point. There is a marked gain in simplicity. It seems probable that automorphic functions are destined to play an important rôle in the theory of algebraic functions and their integrals.

44. Whittaker's Group.—We turn now to the question of actually effecting the uniformisation of algebraic curves by means of functions connected with specific groups. Groups of the type discussed in this section were introduced by Whittaker.*

The groups are constructed from transformations S_i which have the property that $S_i^2 = 1$ (called *self-inverse* transformations). Let $S_1, S_2, S_3, S_4, S_5, S_6$ be real self-inverse transformations satis-

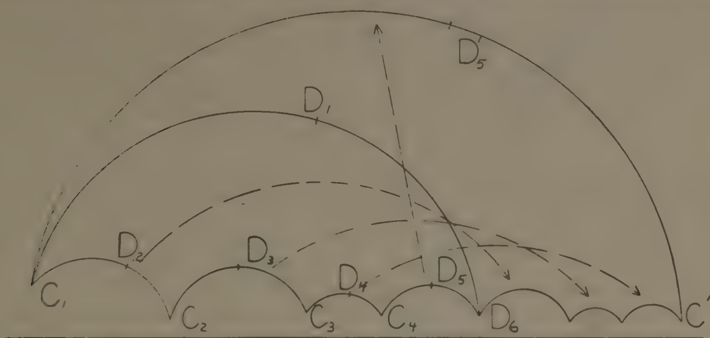

FIG. 29.

fying the equation $S_6S_5S_4S_3S_2S_1 = 1$. This equality gives much freedom of choice, and it is not difficult to determine such transformations.† The transformations are elliptic in type, the fixed points being conjugate imaginaries. Let $D_1, \ldots, D_6$ be the fixed points lying in the upper half-plane. Consider the transforms of the point D_6 when the transformations $S_1, \ldots, S_5$ are made in succession. Since $S_1^2 = 1$, S_1 rotates a short line through D_1 through an angle π. Hence the circle orthogonal to the real axis through D_1D_6 is transformed into itself, D_6 being transformed into a point C_1 on the opposite side of D_1 (see fig. 29). Similarly S_2 carries C_1 into a point C_2, the circle $C_1D_2C_2$ being orthogonal to the axis. Continuing, we get C_3, C_4, C_5. But

* *Phil. Trans.*, **192** (1899), pp. 1–32.

† Whittaker gives as an example the transformations—
$[z, (5z-74)/(z-5)], [z, (2z-5)/(z-2)], [z, (5z-29)/(z-5)], [z, (253z-2061)/(33z-253)],$
$[z, (132z-1675)/(11z-132)], [z, (281z-4786)/(17z-281)]$.

$S_5S_4S_3S_2S_1 = S_6^{-1} = S_6$, a transformation which leaves the point D_6 unchanged. Hence C_5 coincides with D_6, and we have the closed circular polygon shown in the figure.

This polygon is in fact the fundamental region for the group whose generating transformations are S_1, S_2, S_3, S_4, S_5. There are five pairs of congruent sides, D_1D_6 and D_1C_1, D_2C_1 and D_2C_2, etc. The generating transformations satisfy the equations $S_1^2=1, \ldots, S_5^2=1$, and $(S_5S_4S_3S_2S_1)^2 = S_6^2 = 1$. From the last equation we see that the sum of the angles at the congruent vertices $D_6, C_1, \ldots, C_4$ is π (Section 17). By bringing the congruent sides together to form a closed surface we see that the genus of the fundamental region is 0.

Now let us consider the automorphic functions connected with this group. It is known that when the genus of the fundamental region is 0 there exist functions having only one pole in the region; and that in terms of any one of these functions all the automorphic functions of the group are expressible rationally. Let x be one of these functions, and we shall suppose for convenience that its pole does not lie on the boundary. [We can place the pole arbitrarily by using a function of the form $(ax+b)/(cx+d)$, which also has but one pole.] According to Theorem E, Section 21, x acquires any given value once, and only once, in the fundamental region. In other words, the function $x(z)$ maps the fundamental region on the whole x-plane.

Let us examine this mapping. Let $e_1, \ldots, e_5$ be the values of x at the points $D_1, \ldots, D_5$; and e_6 its value at the congruent points $D_6, C_1, \ldots, C_4$. We shall start with the point D_6, and, tracing the boundary of the fundamental region in a counter-clockwise direction, find the corresponding locus in the x-plane. As z moves from D_6 to D_1 x moves along some curve from e_6 to e_1. Now D_1 constitutes a cycle of angle $2\pi/2$; and, according to Theorem A of Section 21, x takes on its value twice there. Hence $x-e_1=(z-D_1)^2\psi(z)$, where $\psi(D_1)\neq 0$, and the angle π at D_1 is mapped on an angle 2π in the x-plane. Since x has the same values along the side C_1D_1 as along D_6D_1, as z moves along D_1C_1 x traverses the curve just found from e_1 back to e_6. C_1 also belongs to a cycle of angle $2\pi/2$, and the angle at C_1 is mapped on an angle twice as great at e_6. The side $C_1D_1C_2$ is mapped on a curve from e_6 to e_2 and back again, and so on. Thus we find that *the function* $x(z)$ *maps the fundamental region on the*

x-*plane, the boundary of the region being mapped on five curves extending from the points* $e_1, \ldots, e_5$ *to the point* e_6 (see fig. 30).

But this group, being of genus 0, will uniformise only the curves of genus 0. We get a group of greater genus in the following manner. Adjoin to the region the region congruent to it by one of the transformations, say S_1. Erase the common side D_6C_1, and the new region is a polygon of eight sides, the opposite sides being congruent by the transformation S_1. The side $C_4D_5D_6$, for example, is transformed into $C_1D_5'C'$ (see fig. 29). The transformation S_5 followed by S_1 transforms $C_4D_5D_6$ into $C'D_5'C_1$, and transforms the new polygon into one adjacent to it along the side $C'D_5'C_1$. A similar treatment of the other sides shows that the double polygon is the fundamental region for the group whose generating transformations are $S_2S_1, S_3S_1, S_4S_1, S_5S_1$. The eight vertices of the polygon form a single cycle of angle 2π. We find that *this region is of genus* 2.

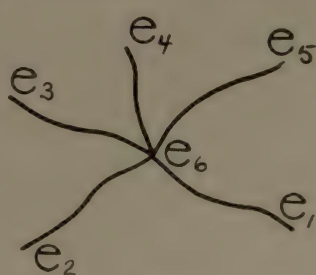

FIG. 30.

This group is a subgroup of the preceding, since all its transformations belong to the former group. Hence $x(z)$ is an automorphic function of the new group; it acquires a given value twice in the double polygon. It maps the polygon on two planes joined together along the branch line e_1e_6. Joining congruent edges of the new fundamental polygon to form a closed region amounts to joining the two x-planes along the remaining branch lines.

Consider a function $w(z)$ automorphic with respect to the new group. The functions w and x are connected by an algebraic relation $\Phi(w, x)=0$, which is of the second degree in w, since to each value of x correspond two points z of the region and hence two values of w. The two-sheeted surface just found is the Riemann's surface for $\Phi(w, x)=0$. Consider in particular the function $w=\sqrt{\{(x-e_1)(x-e_2)(x-e_3)(x-e_4)(x-e_5)(x-e_6)\}}$. This function is single-valued on the surface just found, and is hence an automorphic function of z with respect to the second group. Hence *the automorphic functions of this group enable us to uniformise the curve*—

$$w^2=(x-e_1)(x-e_2)(x-e_3)(x-e_4)(x-e_5)(x-e_6).$$

We can show by means of Theorem G, Section 21, that the general automorphic functions of the group are rational functions of w and x.

The values $e_1, \ldots, e_6$ depend upon the function $x(z)$ and upon the group, and are not all at our disposal. The question that presents itself is: If the function $w^2=(x-e_1)\ldots(x-e_6)$ be arbitrarily chosen, does there exist a group of the type we have been discussing whose automorphic functions effect its uniformisation? As an indication that the uniformisation can always be made by these groups, it will be shown that the constants at our disposal are equal in number to the conditions to be fulfilled.

Each of the transformations $S_1, \ldots, S_6$ contains three essential real constants. The condition $S_i=S_i^{-1}$ imposes one condition on each [namely, that $a=-d$ (Section 1)], leaving twelve constants. The relation $S_6S_5\ldots S_1=1$ defines three of these in terms of the rest. Also this group is not essentially different from one which is obtained by transforming it by any real transformation, which shows that three more of the constants are non-essential. So there are altogether six essential real constants at our disposal in forming the group.

Now considering the x-plane, there are six points $e_1, \ldots, e_6$; and each of these is defined by two real coordinates, giving twelve real constants. But if we make a linear transformation of the x-plane, we can transform any three points into the points e_1, e_2, e_3; which shows that six of the constants can be disregarded as non-essential. So we have six essential constants in the x-figure. Hence, the number of essential constants is the same in the z-figure as in the x-figure, which shows that the groups have the generality requisite for the uniformisation.

It is known that any algebraic relation of genus 2 can be transformed birationally into a relation of the form $w^2=(x-e_1)\ldots(x-e_6)$, so that *the groups in question can be applied to uniformise any algebraic relation of genus* 2.

These groups can be generalised without difficulty. By considering $2p+2$ self-inverse transformations and forming the double polygon as before, we should be able to uniformise the relation

$$w^2=(x-e_1)\ldots(x-e_{2p+2}),$$

which is of genus p.

45. Weber's Group.—In the group employed by Weber * for purposes of uniformisation an initial region bounded by non-intersecting circles orthogonal to the real axis is considered. For simplicity we shall take the case of three circles (C_1, C_2, C_3, fig. 31); the region to be considered lies outside the circles.

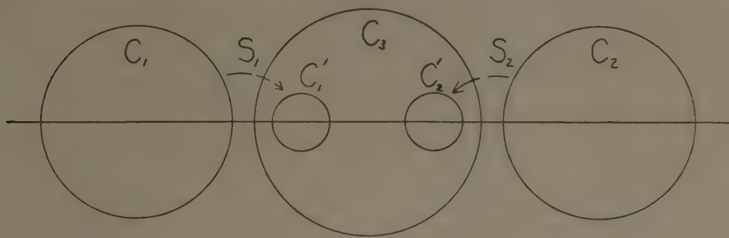

FIG. 31.

From Schwarz's researches in the general theory of conformal representation † Weber shows that there is a function $x(z)$ with the following properties:—

(1) x is analytic within the given region except at infinity, where it has a simple pole; and is continuous on the boundary;

(2) x is real on the circumferences of the circles;

(3) x acquires each value once in the region, and only once save for points on the boundary.

It follows that the function $x(z)$ maps the region bounded by the circles on the whole x-plane. Let us see what the boundary becomes. Since x is real along a bounding circle, as z traces the circle x moves along the real axis, returning to the initial value on the completion of the circuit. *The given region is thus mapped on the* x-*plane bounded by three slits* e_1e_2, e_3e_4, e_5e_6. (See fig. 32.) [It is easy to see that x can trace each slit only twice—once in each direction—and that the slits cannot overlap without violating property (3) for points near the bounding circles.]

FIG. 32.

Now let us add to the given region the region arising by an inversion in the circle C_3. By a well-known theorem on

* *Ein Beitrag zur Poincaré's Theorie der Fuchs'chen Functionen*, Göttinger Nachrichten, 1886, pp. 359–370.

† *Monatsberichten der Berliner Akademie*, 1870.

harmonic continuation * a function which is real along a circle will, if continued analytically across, have the same real part but an imaginary part of opposite sign at points inverse with respect to the circle. The circles C_1, C_2 invert into C_1', C_2' (fig. 31), along which the function x has its values unchanged. If we invert in one of these circles, the imaginary part of x changes sign again, and the result of the two inversions is that the value of x is unchanged. Now, an inversion in C_3 followed by an inversion in C_1' is a linear transformation, S_1, say, carrying C_1 into C_1'. Similarly inversions in C_3 and C_2' are equivalent to a linear transformation S_2 carrying C_2 into C_2'. S_1 and S_2 are generating transformations of a group with the real axis as fixed circle whose fundamental region is the combined region just found. And x is an automorphic function with respect to this group, which acquires each value twice in the region—once in each of the combined portions. The function $x(z)$ maps the region on two planes joined along the branch line e_3e_4 corresponding to C_3. If we bring congruent edges of the region together, we get a closed surface with two holes through it; and the genus of the region is thus 2. Closing the region corresponds in the x-plane to joining the two sheets along the remaining branch lines.

We see, exactly as in the preceding section, that if

$$w^2=(x-e_1)(x-e_2)(x-e_3)(x-e_4)(x-e_5)(x-e_6)$$

w is an automorphic function of z. Hence *the automorphic functions of the group enable us to uniformise this algebraic curve of genus* 2.

We observe that the quantities $e_1, \ldots, e_6$ are real; and we can show by the count of constants that this group is sufficiently general to uniformise any algebraic relation of the form $w^2=(x-e_1)\ldots(x-e_6)$, where $e_1, \ldots, e_6$ are real. As in the preceding case, only three of these quantities are essential. Since the quantities are real, this imposes three conditions on the group.

The group is formed from three circles with their centres on the real axis. Each circle has two degrees of freedom; its centre and its radius may be chosen arbitrarily, save for certain inequalities necessary to ensure that the circles be exterior to one

* Osgood, *Lehrbuch der Funktionentheorie*, vol. i., 2nd ed., pp. 666 *et seq.*

another. There are thus six real constants at our disposal. But three of these can be regarded as non-essential since the group is not essentially different when subjected to a linear transformation. We are left with three essential real constants for the fulfilling of the three conditions imposed, which shows that the groups have the requisite generality to effect the uniformisation.

We have made no use of the fact that the circles are orthogonal to the real axis except to conclude that the group is Fuchsian. [Certain other results that we have not mentioned do follow from this fact; for example, that x has the values $e_1, \ldots, e_6$, at the points where the circles intersect the axis.] If we remove the condition of orthogonality, we get a Kleinian group of the Schottky type (to be mentioned in the next section), and the automorphic functions of this group effect the uniformisation of the relation $w^2=(x-e_1)\ \ldots\ (x-e_6)$, where $e_1, \ldots, e_6$ are real.

46. The Work of Schottky and Burnside.—In an article in *Crelle's Journal*,* Schottky deals with groups derived in the manner of Weber's, but where the circles are not restricted to those orthogonal to a circle. Each circle has an added degree of freedom since its centre is not restricted to lie on the axis, which shows that we have three more constants at our disposal in groups of the Schottky type than in those of Weber (for the case of three circles). There are thus ∞^3 groups of this type which can be used for the uniformisation of the function $w^2=(x-e_1)\ \ldots\ (x-e_6)$, where $e_1, \ldots, e_6$ are real.

Schottky makes a study of the automorphic functions and the Abelian integrals connected with the groups, and obtains infinite products for their expression.

Similar groups are studied by Burnside,† who makes the discovery that in a large class of groups, including those of Weber, the Poincaré theta series (Section 22) converge for $m=1$. It is then possible to get the Abelian integrals by integrating these series. Burnside sets up the various types of integrals in this way, and derives their properties in an exceedingly simple manner.

* Vol. ci. (1887), pp. 227–272. : *Ueber eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Argumentes unverändert bleibt.*

† *On a Class of Automorphic Functions*, Proceedings of the London Mathematical Society, vol. xxiii. (1892), pp. 49–88.

A BIBLIOGRAPHY OF AUTOMORPHIC FUNCTIONS

NOTE.—Many particular classes of functions, *e.g.* the circular functions, elliptic functions, and triangle-functions (including the elliptic-modular and polyhedral functions), had been studied extensively for many years before the general theory of automorphic functions was created. It seems undesirable to attempt here a bibliography of these earlier researches, and the present list will therefore begin with the papers in which Poincaré created the general theory. Mention should, however, be made of one or two memoirs of the earlier period in which the general theory was notably foreshadowed; especially—

SCHWARZ, H. A., *Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elementes darstellt*, Crelle's J., **75** (1872), p. 292.

SCHOTTKY, F., *Über die conforme Abbildung mehrfach zusammenhängender ebener Flächen*, Crelle's J., **83** (1877), p. 300.

FUCHS, L., *Über eine Klasse von Funktionen mehrerer Variabeln, welche durch Umkehrung der Integrale von Lösungen der linearen Differentialgleichungen mit rationalen Coeffizienten entstehen*, Crelle's J., **89** (1880), p. 150.

POINCARÉ, H., *Sur les fonctions fuchsiennes*, C.R., **92** (1881), 333, 393, 859, 1198, 1274, 1484; **93** (1881), 301, 581.

POINCARÉ, H., *Sur les fonctions uniformes qui se reproduisent par des substitutions linéaires*, Math. Ann., **19** (1882), 553; **20** (1882), 52.

KLEIN, F., *Über eindeutige Functionen mit linearen Transformation in sich*, Math. Ann., **19** (1882), 565; **20** (1882), 49.

POINCARÉ, H., *Sur les fonctions fuchsiennes*, C.R., **94** (1882), 163, 1038, 1166; **95** (1882), 626; **97** (1883), 1485.

POINCARÉ, H., *Théorie des groupes fuchsiens*, Acta Math., **1** (1882), 1.

HURWITZ, A., *Ueber eine Reihe neuer Functionen, welche die absoluten Invarianten gewisser Gruppen ganzzahliger linearer Transformationen bilden*, Math. Ann., **20** (1882), 125.

RAUSENBERGER, O., *Ueber eindeutige Functionen mit mehreren nicht vertauschbaren Perioden*, Math. Ann., **20** (1882) 187; **21** (1883), p. 59.

SCHOTTKY, F., *Ueber eindeutige Functionen mit linearen Transformationen in sich*, Math. Ann., **20** (1882), 293.

FUCHS, L., *Ueber Functionen, welche durch lineare Substitutionen unverändert bleiben*, Gött. Nach. (1882), 81.

POINCARÉ, H., *Mémoire sur les fonctions fuchsiennes*, Acta Math., **1** (1883), 193.

PICARD, E., *Sur une classe de groupes discontinus de substitutions linéaires et sur les fonctions de deux variables indépendantes restant invariables par ces substitutions*, Acta Math., **1** (1882), 297.

Poincaré, H., *Sur un théorème de la théorie générale des fonctions*, Bull. de la Soc. Math. de France, **11** (1883) 112.

Poincaré, H., *Mémoire sur les groupes kleinéens*, Acta Math., **3** (1883), 49.

Dyck, W., *Ueber die durch Gruppen linearer Transformationen gegebenen regulären Gebietseinteilungen des Raumes*, Leipz. Ber. (1883), 61.

Klein, F., *Neue Beiträge zur Riemann'schen Functionentheorie*, Math. Ann., **21** (1883), 141.

Picard, E., *Mémoire sur les formes quadratiques binaires indéfinies*, Ann. de l'Éc. Norm. (3), I. (1884), 9.

Poincaré, H., *Sur les groupes des équations linéaires*, Acta Math., **4** (1884), 201.

Picard, E., *Sur un groupe des transformations des points de l'espace situés du même côté d'un plan*, S.M.F. Bull., **12** (1884), 43.

Poincaré, H., *Mémoire sur les fonctions zétafuchsiennes*, Acta Math., **5** (1884), 209.

Picard, E., *Sur les formes quadratiques ternaires indéfinies à indéterminées conjugées et sur les fonctions hyperfuchsiennes correspondantes*, Acta Math., **5** (1884), 121.

Poincaré, H., *Sur un théorème de M. Fuchs*, Acta Math., **7** (1885), 1.

Biermann, O., *Zur Theorie der Fuchs'schen Functionen*, Wien Sitzungsb., **92** (1885), 1137.

von Mangoldt, H., *Ueber ein Verfahren zur Darstellung elliptischer Modulfunctionen durch unendliche Producte nebst einer Ausdehnung dieses Verfahrens auf allgemeinere Functionen*, Gött. Nach. (1886), 1.

Weber, H., *Ein Beitrag zu Poincaré's Theorie der Fuchs'schen Functionen*, Gött. Nach. (1886), 359.

Poincaré, H., *Sur une transformation des fonctions fuchsiennes et la réduction des intégrales abéliennes*, C.R., **102** (1886), 41.

Poincaré, H., *Sur les fonctions fuchsiennes et les formes quadratiques ternaires indéfinies*, C.R., **102** (1886), 735.

Humbert, G., *Application de la théorie des fonctions fuchsiennes à l'étude des courbes algébriques*, Liouville's J. (4), **2**, 239.

Schottky, F., *Ueber eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Argumentes unverändert bleibt*, Crelle's J., **101** (1887), 227.

Picard, E., *Démonstration d'un théorème général sur les fonctions uniformes liées par une relation algébrique*, Acta Math., **11** (1887), 1.

Humbert, G., *Sur le théorème d'Abel et quelques-unes de ses applications géométriques*, Liouville's J. (4), **3** (1887), 327.

Poincaré, H., *Les fonctions fuchsiennes et l'arithmétique*, Liouville's J. (4), **3** (1887), 405.

Stouff, X., *Sur la transformation des fonctions fuchsiennes*, Ann. de l'Éc. Norm. (3), **5** (1888), 219.

Stahl, H., *Ueber die Darstellung der eindeutigen Functionen, die sich durch lineare Substitutionen reproduciren, durch unendliche Producte*, Math. Ann., **33** (1888), 291.

Bianchi, L., *Sulle superficie Fuchsiane*, Lincei Rend. (4), 4_2 (1888), 161.

Picard, E., *Sur les formes quadratiques binaires à indéterminées conjuguées et les fonctions fuchsiennes*, Am. J. Math., **11** (1888), 187.

Schlesinger, L., *Zur Theorie der Fuchs'schen Functionen*, Crelle's J., **105** (1889), 181.

de Brun, F., *Bevis för några teorem af Poincaré*, Öfv. af K. Svenska Vet.-Ak. Förh., **46** (1889), 677.

Hurwitz, A., *Ueber die Differentialgleichungen dritter Ordnung, welchen die Formen mit linearen Transformationen in sich genügen*, Math. Ann., **33** (1889), 345.

Stouff, X., *Sur certains groupes fuchsiens formés avec les racines d'équations binômes*, Toulouse Ann., **4** (1890), P.

Fricke, R., *Ueber eine besondere Klasse diskontinuierlicher Gruppen reeller linearer Substitutionen*, Math. Ann., **38** (1890), 50, 461.

Klein, F., *Zur Theorie der Abel'schen Functionen*, Math. Ann., **36** (1890), p. 1.

Biermann, O., *Ueber die Darstellung der Fuchs'schen Functionen erster Familie durch unendliche Producte*, Monatshefte f. M., **1** (1890), 49, **3** (1892), 143.

Klein, F., *Zur Theorie der Laméschen Functionen*, Gött. Nach. (1890), 85.

Klein, F., *Ueber Normirung der linearen Differentialgleichungen zweiter Ordnung*, Math. Ann., **38** (1890), 144.

Cassel, G., *Öfver en afhandling af H. Weber med titel: "Ein Beitrag zu Poincaré's Theorie der Fuchs'schen Functionen,"* Bihang till Sv. Ak., **16** (1891), Afd. I., No. 2.

Cassel, G., *Sur un problème de représentation conforme*, Acta Math., **15** (1891), 33.

Painlevé, P., *Mémoire sur les équations différentialles du premier ordre*, Annales de l'Éc. Norm. (3), VIII. (1891), 9, 103, 201, 267.

Burnside, W., *On a class of automorphic functions*, Proc. L.M.S., **23** (1892), 49, 281.

Ritter, E., *Die eindeutigen automorphen Formen vom Geschlecht Null*, Gött. Nach. (1892), 283, Math. Ann., **41** (1892), 1.

Klein, F., *Ueber den Begriff des functionentheoretischen Fundamentalbereichs*, Math. Ann., **40** (1892), 130.

Fricke, R., *Weitere Untersuchungen über automorphe Gruppen solcher linearen Substitutionen einer Variabeln, deren Substitutionskoeffizienten Quadratwurzeln ganzer Zahlen enthalten*, Math. Ann., **39** (1891), 62.

Fricke, R., *Ueber discontinuirliche Gruppen, deren Substitutionscoefficienten ganze Zahlen eines biquadratischen Körpers sind*, Gött. Nach. (1892), 268.

Fricke, R., *Ueber ein allgemeines arithmetisch-gruppentheoretisches Princip in der Theorie der automorphen Functionen*, Gött. Nach. (1892), 453.

Schlesinger, L., *Sur la théorie des fonctions fuchsiennes*, C.R., **114** (1892), 1100, 1409.

Schlesinger, L., *Ueber die bei den linearen Differentialgleichungen zweiter Ordnung aufbietenden Primformen*, Crelle's J., **110** (1892), 130.

Burckhardt, H., *Ueber die Darstellung einiger Fälle der automorphen Primformen durch specielle Thetareihen*, Math. Ann., **42** (1893), 185.

del Pezzo, P., *Sui gruppi Kleineani a due variabili*, Napoli Rend. (2), **7** (1893), 123.

Ritter, E., *Die automorphen Formen von beliebigem Geschlechte*, Gött. Nach. 1893), 121.

Fricke, R., *Zur gruppentheoretischen Grundlegung der automorphen Functionen*, Math. Ann., **42** (1893), 564.

Fricke, R., *Ueber indefinite quadratische Formen mit drei und vier Veränderlichen*, Gött. Nach. (1893), 705.

Picard, E., *De l'équation* $\Delta u = ke^u$ *sur une surface de Riemann fermée*, Liouville's J. (4), **9** (1893), 273.

Fricke, R., *Entwickelungen zur Transformation fünfter und siebenter Ordnung einiger specieller automorpher Functionen*, Acta Math., **17** (1893), 345.

Stäckel, P., *Ueber algebraische Gleichungen zwischen eindeutigen Functionen, welche lineare Substitutionen in sich zulassen*, Crelle's J., **112** (1893) 287.

Burnside, W., *Note on linear substitutions*, Mess. (2), **22** (1893), 190.

BURNSIDE, W., *Note on the equation $y^2=x(x^4-1)$*, Proc. L.M.S., **24** (1893), 17.

SCHOENFLIES, A., *Ueber Kreisbogenpolygone*, Math. Ann., **42** (1893) 377.

PICK, G., *Ueber das Formensystem eines Kreisbogenpolygons vom Geschlecht Null*, Math. Ann., **42** (1893), 489.

BIANCHI, L., *Sopra alcune classi di gruppi di sostituzioni lineari a coefficienti complessi*, Math. Ann., **43** (1893), 101.

KAPTEYN, W., *Een paar stellingen van Poincaré*, Handel. v. h. 4de Nederl. Nat. Congres (1893), 149.

FORSYTH, A. R., *Theory of Functions of a complex variable*, Cambridge (1893), (second edition, 1900).

SCHOENFLIES, A., *Ueber Kreisbogendreiecke und Kreisbogenvierecke*, Math. Ann., **44** (1894), 105.

D'OVIDIO. E., *Sulle funzioni Thetafuchsiane*, Torino Atti, **29** (1894), 741

RITTER, E., *Die multiplicativen Formen auf algebraischen Gebilde beliebigen Geschlechtes, mit Anwendung auf die Theorie der automorphen Formen*, Math. Ann., **44** (1894), 261.

FRICKE, R., *Ueber die Transformationstheorie der automorphen Functionen*, Math. Ann., **44** (1894), 97.

RITTER, E., *Die Stetigkeit der automorphen Functionen bei stetiger Abänderung des Fundamentalbereichs*, Math. Ann., **45** (1894), 473, **46** (1895), 200.

FRICKE, R., *Eine Anwendung der Idealtheorie auf die Substitutionsgruppen der automorphen Functionen*, Gött. Nach. (1894), 106.

STOUFF, X., *Sur différents points de la théorie des fonctions fuchsiennes*, Toulouse Ann., viii. (1894) D.

CASSEL, G., *Kritiska studier öfver teorin för de automorfa funktionerna*, Diss. Upsala (1894).

FRICKE, R., *Die Kreisbogenvierseite und das Princip der Symmetrie*, Math. Ann., **44** (1894), 565.

KLEIN, F., *Ueber lineare Differentialgleichungen der zweiten Ordnung*, Autograph. Vorlesung, Göttingen, 1894: Math. Ann., **46** (1895), 77.

FRICKE, R., *Ueber die Discontinuitätsbereiche der Gruppen reeller linearen Substitutionen einer complexen Variabeln*, Gött. Nach. (1895), 360.

BAKER, H. F., *On a certain automorphic function*, Cambr. Proc., **8** (1895), 322.

RITTER, E., *Ueber Riemann'sche Formenschaaren auf einem beliebigen algebraischen Gebilde*, Math. Ann., **47** (1895), 157.

GORDAN, P., *Ueber unverzweigte lineare Differentialgleichungen der zweiten Ordnung auf ebenen Curven vierten Grades*, Math. Ann., **46** (1895), 606.

BAGNERA, F., *Sul teorema dell' esistenza delle funzioni Fuchsiane*, Rivista di Mat., **5** (1896), 31.

KEMPINSKI, S., *Ueber Fuchs'sche Functionen zweier Variabeln*, Math. Ann., **47** (1896), 573: Bull. intern. de Cracovie (1895), 288.

FRICKE, R., *Notiz über die Discontinuität gewisser Collineationsgruppen*, Math. Ann., **47** (1896), 557

FRICKE, R., *Ueber die Theorie der automorphen Modulgruppen*, Gött. Nach. (1896), 91.

FRICKE, R., *Die Theorie der automorphen Functionen und die Arithmetik*, Papers published by the Amer. Math. Soc., I. (1896), 72.

SCHLESINGER, L., *Handbuch der Theorie der linearen Differentialgleichungen*, 2 Bde., Leipzig (1896).

VITERBI, A., *Le equazioni differenziali lineari a coefficienti algebrici integrabili*

algebricamente, studiate in base alla teoria delle "funzione fuchsiane" del Poincaré, Giornale di Batt., **35** (1897), 150 ; **36** (1898), 55.

FRICKE, R., and KLEIN, F., *Vorlesungen über die Theorie der automorphen Functionen*, Leipzig, Teubner, 1897–1912.

FRICKE, R., *Ueber den arithmetischen Charakter gewisser Netze von unendlich vielen congruenten Vierecken*, Braunschw. Festschr. (1897), 85.

BAKER, H. F., *Abel's theorem and the allied theory*, Cambridge (1897).

FRICKE, R., *Die Discontinuitätsbereiche der Gruppen reeller linearer Substitutionen einer complexen Veränderlichen*, Deutsche Math. Ver., **4** (1897), 151.

KLUYVER, J. C., *A special case of Dirichlet's problem for two dimensions*, Acta Math., **21** (1897), 265.

POINCARÉ, H., *Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré*, Palermo Rend., **11** (1897), 193.

POINCARÉ, H., *Les fonctions fuchsiennes et l'équation* $\Delta u = e^u$, Liouville's J., (5), **4** (1898), 137 ; C.R., **126** (1898), 627.

SCHLESINGER, L., *Sur un problème de Riemann*, C.R., **126** (1898), 723.

FRICKE, R., *Ueber die Beziehungen zwischen der Zahlentheorie und der Theorie der automorphen Functionen*, Jahresb. d. Deutsch. Math. Ver., **6** (1898), 94.

WHITTAKER, E. T., *On the connexion of algebraic functions with automorphic functions*, Phil. Trans., **192 A** (1898), 1.

BAGNERA, G., *Un teorema relativo agli invarianti delle sostituzioni di un gruppo Kleinieno*, Rendic. d. Lincei (5ª), **7** (1898), 340.

DIXON, A. C., *Notes on the theory of automorphic functions*, Proc. L.M.S., **31** (1899), 297.

LINDEMANN, F., *Zur Theorie der automorphen Functionen*, Münch. Sitzungsber., **29** (1899), 423 : **30** (1900), 493.

WIRTINGER, W., *Zur Theorie der automorphen Functionen von* n *Veränderlichen*, Wien Ber., **108** (1899), 1239.

FRICKE, R., *Zur Theorie der Poincaré'schen Reihen*, Jahresb. d. Deutschen M. Ver., **9** (1900), 78.

FRICKE, R., *Die automorphen Elementarformen*, Gött. Nach. (1900), 303.

FRICKE, R., *Die Ritter'sche Primform auf einer beliebigen Riemann'schen Fläche*, Gött. Nach. (1900), 314.

WELLSTEIN, J., *Ueber Primformen auf Riemann'schen Flächen*, Gött. Nach. (1900), 380.

DIXON, A. C., *Notes on the theory of automorphic functions*, Proc. L.M.S., **32** (1900), 353.

DIXON, A. C., *Prime functions on a Riemann surface*, Proc. L.M.S., **33** (1900), 10.

HILBERT, D., *Mathematische Probleme*, Gött. Nach. (1900), 253 ; Arch. d. M. u. P. (3), **1** (1901), 44, 213.

FRICKE, R., *Ueber die Poincaré'schen Reihen der* (-1)*-ten Dimension*, Dedekind Festschrift (1901).

HUTCHINSON, J. I., *On a class of automorphic functions*, Trans. Amer. M.S., **3** (1902), 1.

ALEZAIS, R., *Sur une classe de fonctions hyperfuchsiennes et sur certaines substitutions linéaires qui s'y rapportent*, Ann. de l'Éc. Norm., (3), **19** (1902), 261.

WHITTAKER, E. T., *Note on a function analogous to Weierstrass' sigma-function*, Mess. Math., **31** (1902), 145.

FRICKE, R., *Ueber die in der Theorie der automorphen Funktionen auftretenden Polygoncontinua*, Gött. Nach. (1903), 331.

POINCARÉ, H., *Sur l'intégration algébrique des équations linéaires et les périodes des intégrales abéliennes*, Liouville's J. (5), **9** (1903), 139.

BLUMENTHAL, O., *Ueber Modulfunktionen von mehreren Veränderlichen*, M. Ann., **56** (1903), 509; **58** (1904), 497.

FUBINI, G., *Sulla teoria delle forme quadratiche Hermitiane e dei sistemi di tali forme*, Atti d. Gioenia (4), **17** (1903).

BLUMENTHAL, O., *Ueber Thetafunktionen und Modulfunktionen mehrerer Veränderlichen*, Deutsch. Math. Ver., **13** (1904), 120.

FUBINI, G., *Sulla funzioni automorfe e l iperfuchsiane di più variabili indipendenti*, Annali di Mat. (3), **10** (1904), 1.

YOUNG, J. W., *On the group of the sign* (0, 3; 2, 4, ∞) *and the functions belonging to it*, Trans. Amer. Math. Soc., **5** (1904), 81.

FUBINI, G., *Sulla teoria dei gruppi discontinui*, Ann. di Mat. (3), **11** (1904), 159.

FRICKE, R., *Beiträge zum Continuitäts beweise der Existenz linear-polymorpher Funktionen auf Riemannschen Flächen*, Math. Ann., **59** (1904), 449.

FUBINI, G., *Una questione fondamentale per la teoria dei gruppi e delle funzioni automorfe*, Rend. Lincei, **13** (1904), 590.

FRICKE, R., *Neue Entwicklungen über den Existenz-beweis der polymorphen Funktionen*, Heidelberg dritt Math. Kongress (1904), 246.

HUTCHINSON, J. I., *On the automorphic functions of the group* (0, 3; 2, 6, 6), Trans. Amer. Math. Soc., **5** (1904), 447.

BLUMENTHAL, O., *Bemerkungen zur Theorie der automorphen Funktionen*, Gött. Nach. (1904), 92.

POINCARÉ, H., *Sur les invariants arithmétiques*, Crelle's Journ., **129** (1905), 89.

HURWITZ, A., *Zur Theorie der automorphen Funktionen von beliebig vielen Variabeln*, Math. Ann., **61** (1905), 325.

MORRIS, R., *On the automorphic functions of the group* (0, 3; l_1, l_2, l_3), Tr. Am. M.S., **7** (1906), 425.

BRODÉN, T., *Zur Theorie der mehrdeutigen automorphen Funktionen*, Lunds Universitets Års-skrift, **40** (1904), 54.

JOHANSSON, S., *Ein Satz über die konforme Abbildung einfach zusammenhängender Riemann'scher Flächen auf den Einheitskreis*, Math. Ann., **62** (1906), 177.

JOHANSSON, S., *Beweis der Existenz linear-polymorpher Funktionen vom Grenzkreistypus auf Riemann'schen Flächen*, Math. Ann., **62** (1906), 184.

HUTCHINSON, J. I., *On automorphic groups whose coefficients are integers in a quadratic field*, Trans. Ann. M.S., **7** (1906), 530.

LEVI, E. E., *Ricerche sulla teoria delle funzioni automorfe*, Atti R. Acc. Lincei (5), **15** (1906), 682.

FUBINI, G., *Sulla teoria delle funzioni automorfe*, Atti R. Inst. Ven., **65** (1905), 1297.

HERGLOTZ, L., *Ueber die Gestalt der auf algebraischen Kurven nirgends singulären linearen Differentialgleichungen zweiter Ordnung*, Math. Ann., **62** (1906), 329.

FUBINI, G., *Sulla costruzione dei compi fondamentali di un gruppo discontinuo*, Ann. d. Mat. (3), **12** (1906), 347.

HUTCHINSON, J. I., *A method for constructing the fundamental region of a discontinuous group of linear transformations*, Tr. Am. M.S., **8** (1907), 261.

KOEBE, P., *Ueber die Uniformisierung reeller algebraischer Kurven*, Gött. Nach. (1907), 177.

KOEBE, P., *Ueber die Uniformisierung beliebiger analytischer Kurven*, Gött. Nach. (1907), 191, 633.

Fubini, G., *Nuove ricerche interno ad alcune classi di gruppi discontinui*, Palermo Rend., **21** (1906), 177.

Fatou, P., *Sur les solutions uniformes de certaines équations fonctionnelles*, C.R., **143** (1906), 546.

Stahl, H., *Berechtigung einer Arbeit von Herrn E. T. Whittaker*, Arch. d. M. u. P. (3), **10** (1906), 336.

Klein, F., *Bemerkungen zur Theorie der linearen Differentialgleichungen zweiter Ordnung*, Math. Ann., **64** (1907), 175.

Pick, G., *Ueber die zu einer ebenen algebraischen Kurve gehörigen transzendenten Formen und Differentialgleichungen*, Monatshefte f. M. u. P., **18** (1907), 219.

Koebe, P., *Zur Uniformisierung der algebraischen Kurven*, Gött. Nach. (1907), 410.

Fubini, G., *Sulla teoria delle funzioni automorfe e delle loro transformazioni*, Ann. di Math. (3), **14** (1907), 33.

Poincaré, H., *Sur l'uniformisation des fonctions analytiques*, Act. Mat., **31** (1907), 1.

Schottky, F., *Ueber Beziehungen zwischen veränderlichen Grössen, die auf gegebene Gebiete beschränkt sind*, Berlin Sitzungsb. (1907), 919 ; and (1908), 119.

Klein, F., *Ueber die Zusammenhang zwischen dem sogenannten Oszillationstheorem der linear Differentialgleichungen und dem Fundamentaltheorem der automorphen Funktionen*, Deutsch. Math. Ver., **16** (1907), 537.

Young, J. W., *On a class of discontinuous ζ-groups defined by normal curves of the fourth order*, Palermo Rend., **23** (1907), 97.

Young, J. W., *A fundamental invariant of the discontinuous ζ-groups defined by the normal curves of order* n *in a space of* n *dimensions*, Bull. Am. M.S. (2), **14** (1908), 363.

Rothe, H., *Ueber das Grundtheorem und die Obertheoreme der automorphen Funktionen im Falle der Hermite-Laméschen Gleichung mit vier singulären Punkten*, Monat. Math. u. Phys., **19** (1908), 258.

Koebe, P., *Die Uniformisierung der algebraischen Kurven. Mitteilung eines Grenzübergangs durch iterierendes Verfahren*, Gött. Nach. (1908), 112.

Hilb, E., *Ueber Kleinsche Theoreme in der Theorie der linearen Differentialgleichungen*, Math. Ann., **66** (1908), 215 ; **68** (1909), 24.

Garnier, R., *Sur les équations différentielles du troisième ordre dont l'intégrale générale est uniforme*, C.R., **147** (1908), 915.

Hilb, E., *Neue Entwicklungen über lineare Differentialgleichungen*, Gött. Nach. (1908), 231 ; (1909), 230.

Poincaré, H., *Sur la réduction des intégrales abéliennes et les fonctions fuchsiennes*, Rend. d. Circolo d. Palermo, **27** (1908), 281.

Caspar, M., *Ueber die Darstellbarkeit der homomorphen Formenscharen durch Poincaré'sche Zetareihen*, Diss. Tübingen (1908).

Fubini, G., *Sulla discontinuità propria dei gruppi discontinui*, Atti iv. congresso. intern. matem. (Rome), vol. ii. (1908), 169.

Fubini, G., *Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe*, Pisa (1908).

Koebe, P., *Ueber die Uniformisierung der algebraischen Kurven*, Math. Ann., **67** (1909), 145.

Richmond, H. W., *On automorphic functions in relation to the general theory of algebraic curves*, Quart. Journ. M., **40** (1909), 258.

Young, J. W., *On the discontinuous ζ-groups defined by rational normal curves in a space of* n *dimensions*, Bull. Am. M.S. (2), **16** (1909), 363.

HILB, E., *Ueber Kleinsche Theoreme in der Theorie der linearen Differentialgleichungen*, Math. Ann., **68** (1910), 24.

POINCARÉ, H., *Sechs Vorträge über ausgewählte Gegenstände aus der reinen Mathematik und mathematischen Physik*, Leipzig (1910).

BIEBERBACH, L., *Zur Theorie der automorphen Funktionen*, Diss. Göttingen (1910).

CRAIG, C. F., *On a class of hyperfuchsian functions*, Trans. Amer. Math. Soc., **11** (1910), 37.

KOEBE, P., *Ueber die Uniformisierung der algebraischen Kurven durch automorphe Funktionen mit imaginärer Substitutionsgruppe*, Gött. Nach. (1909), 68; (1910), 180.

KOEBE, P., *Ueber die Uniformisierung der algebraischen Kurven*, Math. Ann., **69** (1910), 1.

KOEBE, P., *Ueber die Uniformisierung beliebiger analytischer Kurven*, Gött. Nach. (1908), 337; (1909), 324.

HILBERT, D., *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Sechste Mitteilung*, Gött. Nach. (1910), 355.

KOEBE, P., *Ueber die Hilbertsche Uniformisierungsmethode*, Gött. Nach. (1910), 59.

HECKE, E., *Ueber die Konstruktion der Klassenkörper reeller quadratischer Körper mit Hilfe von automorphen Funktionen*, Gött. Nach. (1910), 619.

PRYM, F., and ROST, G., *Theorie der Prymschen Funktionen erster Ordnung*, Leipzig (1911).

COURANT, R., *Ueber die Anwendung des Dirichletschen Prinzipes auf die Probleme der konformen Abbildung*, Math. Ann., **71** (1911), 145.

CHAZY, J., *Sur l'indétermination des fonctions uniformes au voisinage de leurs coupures*, C.R., **152** (1911), 499.

LILJESTRÖM, A., *Sur les fonctions fuchsiennes à multiplicateurs constants*, Arkiv Matem. Astr. Fys., **6** (1910), 18.

KOEBE, P., *Ueber die Uniformisierung beliebiger analytischer Kurven*, Crelle's J., **138** (1910), 192; **139** (1911), 251.

HECHE, E., *Höhere Modulfunktionen und ihre Anwendung auf die Zahlentheorie*, Math. Ann. **71** (1911), 1.

FRICKE, R., *Zur Transformationstheorie der automorphen Funktionen*, Gött. Nach. (1911), 518.

BIANCHI, L., *Sul gruppo automorfo delle forme ternarie quadratiche suscettibili di rappresentare lo zero*, Atti Accad. Lincei (5) **21** (1912), 305.

JOHANSSON, S., *Zur theorie der Konvergenz der Poincaré'schen Reihen bei den Hauptkreisgruppen*, Öfversigt af Finska Vetenskaps-Soc. Förhand., **53** (1910), 15, p. 25.

JOHANSSON, S., *Om konvergensen af de Poincaré'ska θ-serierna i hufvudcirkelfallet*, 2d Congr. math. Scand. (1911), 181.

FRICKE, R., *Zur Transformationstheorie der automorphen Funktionen*, Gött. Nach. (1912), 240.

BIEBERBACH, L., $\Delta u = e^u$ *und die automorphen Funktionen*, Gött. Nach. (1912), 599.

BROUWER, L. E. J., *Ueber die topologischen Schwierigkeiten des Kontinuitäts beweises der Existenztheoreme eindeutig umkehrbarer polymorpher Funktionen auf Riemannschen Flächen*, Gott. Nach. (1912), 603.

OSGOOD, W. F., *Lehrbuch der Funktionentheorie*, Leipzig (2nd edition) (1912).

BROUWER, L. E. J., *Ueber die Singularitätenfreiheit der Modulmannigfaltigkeit*, Gött. Nach. (1912), 803.

Klein, F., Brouwer, L. E. J., Koebe, P., Bieberbach, L., und Hilb, E., *Zu den Verhandlungen betreffend automorphen Funktionen*, Deut. Math. Ver., **21** (1912), 153.

Koebe, P., *Ueber die Uniformisierung der algebrai Kurven*, Math. Ann., **72** (1912), 437.

Koebe, P., *Zur Begründung der Kontinuitätsmethode*, Leip. Ber., **64** (1912), 59.

Garnier, R., *Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme, et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes*, Ann. de l'école supér. (3), **29** (1912), 1.

Liljeström, A., *Études sur la théorie du potentiel logarithmique*, Arkiv Matem. Astro. Fys., **7** (1912), 39, p. 56.

Plemelj, J., *Die Grenzkreisuniformisierung analytischer Gebilde*, Monats. f. M. u. P., **23** (1912), 297.

Koebe, P., *Begründung der Kontinuitätsmethode im Gebiete der konformen Abbildung und Uniformisierung*, Gött. Nach. (1912), 879.

Osgood, W. F., *Existenzbeweis betreffend Funktionen, welche zu einer eigentlich discontinuierlichen automorphen Gruppe gehören*, Palermo Rend., **35** (1913), 103.

Osgood, W. F., *On the uniformisation of algebraic functions*, Annal. Math. Princeton (2) **14** (1913), 143.

Fricke, R., *Automorphe Funktionen mit Einschluss der elliptischen Modulfunktionen*, Encyclopädie der math. Wiss., Band ii. 2, Leipzig (1913).

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